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APPLICATION OF DIFFERENTIAL GAMES

TO PROBLEMS OF MILITARY CONFLICT:

TACTICAL ALLOCATION PROBLEMS - PART III

James G. Taylor

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The mathematical theory of differential games is used to study the structure of optimal allocation strategies for some time-sequential combat games with combat described by Lanchester-type equations of warfare. As required by such applications, some new theoretical results are given: first order necessary conditions of optimality are developed for differential games with state variable inequality constraints. These results are used to study optimal air-war strategies in a differential game model. In a different

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model, optimal air-war strategies are further studied within the context of land-war objectives. Optimal fire-support strategies are studied in an attack scenario with a differential game model. A comprehensive survey of previous literature on each of the above topics is given. Finally, some problems for possible future study are discussed.

SUMMARY

The mathematical theory of differential games is used to study the structure of optimal allocation strategies for some time-sequential combat games with combat described by Lanchester-type equations of warfare. Although most of this work concerns the application of existing theory for the determination of optimal campaign strategies in various Lanchester-type differential games of tactical interest, theoretical developments are also given consideration.

Previous research by the author had uncovered an important gap in the existing theory of differential games: no adequate theory of state variable inequality constraints existed for differential games. Such a theory is essential for Lanchester-type differential games because, for example, of the requirement that force levels be non-negative. Consequently, in this report first-order necessary conditions of optimality are developed for differential games with state variable inequality constraints. These results are then applied to the study of optimal compaign strategies in several differential games of tactical interest.

Optimal air-war campaign strategies are studied in a generalization of a well-known tactical air-war model. Previous work had never given adequate consideration to necessary conditions of optimality with respect to the non-negativity of aircraft force levels. The work at hand extends previous work by considering a value for survivors at the end of the air war and variable (effectiveness) coefficients. Several specific problems are solved. The effects of temporal variations in the return from ground support on optimal campaign strategies are studied.

The differential game model discussed above has been criticized because it does not evaluate air-war tactics within the context of ground-war objectives. Consequently, this model produces suboptimization. Hence, a model is considered which is considerably broader in scope, does consider land-war objectives, but (unfortunately) is considerably less susceptible to closed-form analytic solution. For this new model partial results concerning the determination of optimal campaign strategies are obtained and contrasted with those for the model previously discussed above.

The determination of optimal fire distribution strategies for supporting weapon systems is a major problem of military operations research. The author has studied some aspects of this problem in previous research. Motivated by a paper which recently appeared in the open literature, optimal fire-support strategies are studied in an attack scenario. The dependence of optimal fire-support strategies on combatant objectives is examined for a problem previously considered in the open literature: necessary and sufficient conditions on the functional form of the criterion functional (terminal payoff only) for the optimal fire-support strategies to be independent of force levels (at least for a certain range of values) are developed. This shows that certain results previously reported in the literature are, in general, not true. A variation of this problem is then considered. In contrast to the previous work reported in the literature, the attrition structure of the problem at hand leads to the optimal fire-support strategy of the attacker requiring him to sometimes split his artillery fire between enemy infantry and artillery (counterbattery fire). Numerical examples are given.

Finally, some problems for possible future study are discussed. The formulation of these problems is based on the author's past research experiences. As is usually the case, work on the problems considered here and elsewhere has uncovered other aspects of interest that appear to merit further examination.

TABLE OF CONTENTS

1. Introduction.

This report is the third in a series of reports which document the author's research on the structure of optimal (time-sequential) allocation policies for tactical situations described by Lanchester-type equations of warfare. By considering several combat scenarios, further insights have been gained into such important questions as:

(1) How should fire be distributed over targets?

(2) How should targets be selected?

(3) Do target priorities change over time?

(4) Do force levels affect the optimal allocation strategies?

(5) How does the number of target types affect the optimal allocation strategies?

(6) Do conflict termination circumstances affect the optimal allocation strategies?

(7) How are the optimal fire distribution/target selection strategies affected by the nature of the attrition processes?

(8) What is the effect of logistics constraints on such policies?

(9) How does the uncertainty and confusion of combat affect optimal allocation strategies?

Specific objectives of this research have been the identification of decision parameters and the further development of general principles for target selection, regulation of firing rate, and allocation of military resources in dynamic combat situations. A further discussion of research objectives and their relationship to defense planning problems can be found in the author's past reports* [11], [12].

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*A comprehensive review of pertinent literature prior to 1973 in the fields of (a) Lanchester-type models of warfare, (b) differential games, and (c) optimal control of Lanchester-type attrition processes is also to be found in [12].

It should be pointed out that although the quantitative analysis of tactics may be considered to be a relatively new field of military operations research, there is wide-spread interest in such work (see [6],[8],[9],[10],[19]).

Our approach has been to combine Lanchester-type models of warfare with generalized control theory (both deterministic and stochastic optimal control, dynamic programming, and differential games) [12],[15] (see also [16]). This has been done by considering a sequence of concrete problems. Additionally, the report at hand contains some basic research on the theory of differential games. Past research [12],[15] had identified as an important gap in the existing theory of differential games the lack of adequate theory (i.e. necessary conditions of optimality) for problems with state variable inequality constraints (SVIC's).

The organization of this report is to discuss results in general terms in the main body and to leave details for the appendices. This has been partially necessitated by the wide scope of problems examined (from theoretical to applied).

2. Guided Tour of the Appendices.

In this section we summarize the work which is contained in the appendices and explain why this work was done. The results reported here may be considered to be extensions of our earlier work [11],[12],[15]. Moreover, the work at hand lays the foundations for more extensive work on the quantitative analysis of tactics and applications of generalized control theory to problems of military operations research.

In Appendix A we develop a theory of state variable inequality constraints for Lanchester-type differential games. First order necessary conditions of optimality are developed for the class of problems in which individual state variables are bounded by functions of time. Although this certainly is not the most general type of state variable inequality constraint (SVIC), it does include all Lanchester-type differential games that occur in military operations research. First order necessary conditions of optimality are also stated (but not proved) for more general problems. In particular, results are given without proof for problems with a fairly general type of SVIC. The proofs of these results will be given in the future.

Recently, we pointed out [12],[15] that no adequate multiplier conditions existed for (Lanchester-type) differential games with state variable inequality constraints and gave this top research priority. It is indeed remarkable that essentially all previous applications of the theory of differential games to Lanchester-type problems (see, for example, [1],[2],[4],[5],[7]) are inadequate in one respect or another with respect to SVIC's. A further discussion and substantiation of this claim is to be found in Appendix A. This work establishes the mathematical foundations for future applications of the theory of differential games because force levels must be, for example, non-negative (see [17] for a discussion within the context of Lanchester-type optimal control problems).

In Appendices B and C we apply the theoretical results of Appendix A to the study of optimal air-war strategies. This has been over the years a problem of lasting interest for defense planners. In Appendix B

we consider a generalization of the tactical air-war game of A. Mengel [4] (see also [3]) in which aircraft effectiveness may vary over time and a residual value for aircraft at the end of the campaign is considered. Optimal strategies are characterized for this general problem. A complete solution (i.e. explicit determination of optimal closed-loop strategies) is given for cases of constant coefficients. These results are much more general than previous ones and provide much insight into tradeoffs between various planning parameters. Preliminary results are also obtained for problems in which aircraft effectiveness in the air war is constant but the value of ground support changes over time. The latter is an attempt to reflect the influence of combat air support on the ground war and hence avoid suboptimization. Two cases of time-dependent "returns" from ground support are considered:

(a) linearly-decreasing-with-time returns from ground support,

(b) exponentially-decreasing-with-time returns from ground support.

One criticism that has been made of the model considered in Appendix B is that it does not evaluate air-war strategies within the context of ground-war objectives. In other words, this model (unfortunately) produces suboptimization. Thus, in Appendix C we present preliminary results for a model which considers the development of optimal air-war strategies within the context of land-war objectives. These preliminary results show that the outcome of the ground war may be a significant factor in the determination of optimal air-war strategies.

The research reported in Appendices B and C was undertaken because an important question for defense planners is what are appropriate missions over the course of a campaign for tactical air power. The answer to this question has far-reaching implications for Navy air forces

(both carrier-based and land-based) and, of course, the Air Force. Recently, the USAF Studies and Analysis Group has been using quantitative methodology [19] in trying to answer such questions. By considering the results given in Appendices B and C one can begin to see the effects of the nature of the combat optimization problem on the structure of optimal air-war strategies.

The determination of optimal fire distribution strategies for supporting weapon systems is a major problem of military operations research. This problem is particularly of interest to the military tactician so that he may have a clearer understanding of the circumstances, for example, under which enemy infantry should be engaged by a supporting weapon system (such as artillery) and those under which "counter-battery" fire is to be preferred. Furthermore, the study of such tactical time-sequential allocation problems is relevant to the Navy mission of fire support (both by ship gunfire and by carrier-based air).

In Appendix D we further consider the determination of optimal fire distribution strategies for supporting weapon systems. In this case it is appropriate to call the supporting weapon system artillery. We consider extensions of some recent work by Kawara [5] concerning optimal strategies for supporting weapon systems in an attack scenario which is a variation of a model considered by Weiss [20]. We first examine for what class of criterion functionals (terminal payoff only) the optimal fire-support strategies are <u>independent</u> of the force levels and then examine the dependence of the optimal fire-support strategies upon the form of the combat attrition model by considering slightly different combat dynamics than those considered by Kawara [5]: we assume that the attacker's artillery

produces "linear-law" attrition (see [13],[16]) against both the defender's artillery as well as infantry. The development of a complete solution to the latter problem has involved solution phenomena not previously encountered in a Lanchester-type differential game: the dual variables are discontinuous across a manifold of discontinuity of both players' strategies.

The research reported in Appendix D was undertaken to obtain a better understanding of the dependence of the structure of optimal fire-support strategies upon model form. Previous work by Weiss [20] (see also [4]) and Kawara [5] had given the impression that an optimal fire-support strategy consists in always concentrating all fire on one enemy target type. From our previous work [13] on a one-sided Lanchester-type, time-sequential fire distribution problem in which each of two enemy target types undergoes a "linear-law" attrition process, we knew that an optimal fire distribution policy could consist in splitting one's fire between available target types. This was our motivation for the examination of other attrition structures in Kawara's problem. Additionally, Kawara's recent work [5] stimulated our research on the influence of combatant objectives on optimal fire-support strategies because of Kawara's remarkable result that (at least within a certain range of force levels) optimal fire-support strategies do not depend on force levels. We were able to show that Kawara had used the only type of objective function that yeilds this result. Thus, in general, the result is not true.

In Appendix E we briefly describe several important problems for possible future research. These problems were formulated giving

consideration to our past research experiences (both that reported here and also elsewhere). These problems may be referred to as

(a) optimal fire support for several ground units,

(b) a Lanchester-type optimal control problem with logistics considerations,

(c) an examination of the influence of the form of the criterion functional on optimal strategies in time-sequential combat optimization problems.

As explained in Appendix E, work on these campaign analysis problems would appreciably enhance our understanding of methodology for the optimization of combat dynamics.

3. Summary of Research Findings.

Here we summarize our research results. We have (at least partially) accomplished the following tasks which were suggested for future research in our previous report [12]: (a), (b2), (e), (g)*, (i1), (j2), and (k). Results are organized under the following headings:

(1) solution methodology for time-sequential combat games,
(2) insights gained into the optimization of combat dynamics,
(3) implications for defense planning.

Items (2) and (3) differ in that the latter is a management-oriented digest of the practical implications of our research, whereas the former is oriented towards a technical audience. Further amplification of results and conclusions is to be found in the appendices.

a. Solution Methodology for Time-Sequential Combat Games.

Our research has produced the following results on solution methodology for time-sequential combat games. Specifically, we have accomplished the following:

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*See [18].

(1) developed theory of state variable inequality constraints for Lanchester-type differential games (This theory is essential for all problems with, for example, negativity restrictions on force levels.),

(2) demonstrated use of theory of state variable inequality constraints for solving time-sequential combat games by developing solutions to several specific combat optimization problems,

(3) developed complete solution to a fire support differential game. This involved the following technical difficulties:

    (a) singular surfaces,
    (b) discontinuity in the dual variables,

(4) developed methodology for determining the functional form of the criterion functional (terminal payoff here) which yields optimal strategies that do not depend on force levels (at least for a certain range of force levels),

(5) concluded that computational methods must give consideration to structural properties of optimal strategies in idealized versions like those considered in this report.

b. Insights Gained into the Optimization of Combat Dynamics.

Based on our study of the optimization of combat dynamics using the mathematical theory of differential games, we have reached the following conclusions:

(1) the structure of optimal strategies depends on the following factors:

    (a) criterion functional,
    (b) combat attrition model,
    (c) battle termination model;

    the dependence is complex, and future research should concentrate upon the examination of numerous structures of tactical interest,

(2) optimal fire-support strategies depend on the nature of the attrition process produced by the supporting weapon system; when the supporting weapon system produces casualties at a rate proportional to the product of the numbers of firers and targets of a particular type (see Appendix D for mathematical formulation), the optimal fire-support strategy of the supporting weapon system has the following characteristics:

(a) it depends on the optimal strategy for the enemy's supporting weapon system,
(b) the optimal distribution of supporting fires may be
    (i) to concentrate on enemy infantry,
    (ii) to split fire to avoid overkilling,
    (iii) to concentrate on enemy artillery,
(c) counterbattery fire is the optimal strategy during the early stages of an attack and destruction of enemy infantry is the optimal strategy during the final moments (unless the ratio of defender infantry to defender artillery is "extremely" large, in which case the infantry is always engaged),
(d) a split of supporting fires between enemy infantry and artillery as an optimal strategy can only occur when enemy infantry have some effectiveness against friendly infantry,

(3) optimal air-war strategies are different for the case in which ground war objectives are considered than for that in which they are not,

(4) optimal air-war strategies (see Appendix B for scenario) depend on the following factors:

(a) residual value of surviving aircraft,
(b) force levels (especially when a side is going to lose the fight for air supremacy: faced with a future loss of airpower, one may abandon the fight for air supremacy and get what ground support one can from his planes before he loses them all),

(5) cases with variations over time in weapon system effectiveness are best studied as extensions of constant coefficient cases.

c. Implications for Defense Planning.

In our research reported here we have studied idealizations of allocation structures that commonly occur in defense planning studies. After studying these idealizations in order to gain insight into the structure of optimal strategies in the complex real-world problem, we have reached the following conclusions as to considerations that should be brought to the attention of defense planners. These results should be kept in mind by practitioners who perform more detailed computer simulation studies.

(1) The combat optimization problem should be thought of as consisting of three parts:
   (i) combatant objectives,
   (ii) conflict termination conditions,
   (iii) combat dynamics.

   Optimal combat strategies depend on all three of the above. More basic scientific research should be done on all three, especially the first two.

(2) Force levels do affect (either directly or indirectly) optimal combat strategies.

(3) It may be quite dangerous to generalize optimal combat strategies developed for specific problems. At present, more research is needed on specific problems in order to develop an understanding of the qualifications that may be necessary to make about specific study results.

(4) Optimal air-war strategies must be based on ground-war objectives. Suboptimization results when this is not done. This suboptimization is a serious problem, since it may lead to winning the air campaign but losing the war.

4. Suggested Future Research Tasks.

After performing the research documented in this report, we feel that the current state-of-the-art for applying the theory of differential games to time-sequential combat games is such that more significant results may be readily obtained in the future. Moreover, this previous research provides valuable perspective for identifying what appear to be the most important research tasks to be considered next. In our opinion the most important task is to examine (and then understand) the influence of objectives on optimal strategies. Another important task is to study the structure of optimal fire-support strategies.

Based on our past research experience we feel that there is much to be accomplished in the future. Specifically, we suggest the following as future research tasks:

(a) Examination of the effects of campaign objectives (as reflected by the criterion functional) on the structure of optimal campaign strategies. In all our past research we have with one exception

always considered a linear utility for survivors in the criterion functional. It is of interest to examine how the valuation of survivors affects the structure of optimal strategies. Other criterion functionals that might be considered are

(1) nonlinear valuation of survivors (either the difference or ratio of functions which are, for example, concave, quasi-concave, etc.),
(2) the value of losses rather than survivors.

It would seem appropriate to begin such an investigation by considering the simplest problem possible.

(b) Further study of optimal fire-support strategies. The actual military operations must be analyzed and decisions identified. For various cases, the appropriate scenario and, consequently, Lanchester-type model would be developed (see Appendix E), and the optimization problem studied. It is felt that at this time it is most important to study various specific problems in order to gain insight into the structure of optimal fire-support strategies.

(c) Examination of the effects of logistics constraints on optimal campaign strategies. Models would be developed to relate logistics capability to combat effectiveness and then appropriate combat optimization problems formulated. Such research would provide insight into the worth of the Navy logistics (pipeline) role in combat service support missions (see Appendix E).

(d) Further study the determination of optimal air-war strategies within the context of land-war objectives. Based on preliminary results given in this report, it appears as though the outcome of the ground war is a significant factor* in the determination of optimal air-war strategies and that optimal strategies developed for a model not considering land-war objectives need not be optimal for a model which does consider them. This proposed research would extend the results given in Appendix C of the report at hand.

A more comprehensive (although somewhat dated now) discussion of suggested future research tasks can also be found on pages 65-67 of our previous report [12]. Work still remains to be done on the following tasks suggested there: (b), (c), (d), (f), (h), and (j).

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*This is not considered, for example, in TAC CONTENDER [19].

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APPENDIX A: Necessary Conditions of Optimality for Differential Games with State Variable Inequality Constraints.

1. Introduction.

In our most recent report on applications of differential games to problems of military conflict [22] we suggested as a future research task that a theory of state variable inequality constraints (SVIC's) be developed for Lanchester-type differential games (see [23]). This suggestion was based on our discovery of an important gap in the existing theory of differential games: no adequate treatment of multiplier conditions for differential games with SVIC's then existed in the literature. As we have pointed out previously [23], [25], state variable inequality constraints are present in all Lanchester-type dynamic tactical allocation problems because force levels (which are represented by state variables) are required to be non-negative (or some equivalent condition). Thus, the development of a theory of SVIC's is essential for an adequate characterization of an optimal strategy in such a problem (see examples given in Appendix B). It is indeed remarkable that this important gap in theoretical results has not even been noted by other workers in operations research.

The origins of the theory of differential games are given in [10]. Since the publication of this paper in 1965 and that of the highly significant book by Isaacs [11], numerous papers have been published on the subject of differential games. Further references can be found in [22]. Necessary conditions of optimality have been developed for differential games by Isaacs [11] and Berkovitz [3], [4]. In two highly technical and significant papers [3], [4], Berkovitz developed necessary conditions for problems

subject to various types of inequality constraints <u>except</u> those only involving functions of the state variables (i.e. except SVIC's). Variational methods were used in [3] by reducing the problem to two associated variational problems. In [4] the properties of the value function were utilized to develop these conditions by use of Isaacs' "tenet of transition." Schmitendorf and Citron [16] extended Berkovitz's variational approach to second order conditions and developed a conjugate point condition for a class of differential games. Later, Schmitendorf [15] used the same approach to develop (both first and also second order) necessary conditions of optimality for (zero-sum deterministic) differential games in which one or both players are restricted to use open-loop control. The development of necessary conditions has also been disucssed by Chattopadhyay [7].

In this appendix we will develop necessary conditions of optimality for differential games with SVIC's by an extension of the approach of Berkovitz [3]. The development of necessary conditions for such problems has apparently not been accomplished before although sufficient conditions for optimality have been derived by a number of workers [13], [18].

Quite independently of the above developments A. Friedman began a careful study of the existence of value for a differential game and existence of a saddle point in pure strategies. This work led to the book [9] and has continued. Differential games with SVIC's have been considered by Friedman, but he has focused primarily on the question of existence of value for them (see Chapter 6 of [9]). We have used Friedman's results in our past work [19], [20], [21]. However, Friedman [9] does not develop the two-sided analogues of well-known control theory multiplier conditions for constrained subarcs which lie on the boundary of the state space [12].

In fact, the treatment of the example on pp. 239-240 of [9] is inadequate, since the approach fails to yield optimal strategies for minor modifications in the problem's formulation (see Appendix B). Moreover, this problem is very important for military operations research (see Appendix B).

The research reported in this appendix is of a more basic nature than that reported elsewhere in this report. These results are also of much wider application than to the problems of defense planning upon which the author has concentrated. Moreover, the derivation of these results allows one to identify the class of problems to which such multiplier conditions apply. It should be noted that not all system dynamics give rise to such conditions.

2. A Lanchester-Type Differential Game.

In [23] we coined the term Lanchester-type differential game as referring to a differential game in which the system dynamics are described by Lanchester-type equations of warfare. To be precise, we are considering two-person zero-sum deterministic differential games in which each player uses a closed-loop (or feedback) strategy. A formulation which is relevant to our research on tactical allocation problems is as follows:

$$\underset{\Psi_{ij}}{\text{maximize}}\ \underset{\Phi_{ij}}{\text{minimize}}\ \left\{ \sum_{i=1}^{n} w_i y_i(t_f) - \sum_{i=1}^{m} v_i x_i(t_f) \right\},$$

stopping rule: $\quad t_f - T = 0,$

subject to: (combat dynamics)

$$\frac{dx_i}{dt} = r_i - \sum_{j=1}^{n} \psi_{ij} a_{ij} y_j \quad \text{for} \quad i = 1,\ldots,m,$$

$$\frac{dy_i}{dt} = s_i - \sum_{j=1}^{m} \phi_{ij} b_{ij} x_j \quad \text{for} \quad i = 1,\ldots,n, \qquad (1)$$

with

$$x_i, y_j \geq 0 \quad \text{for all} \quad i,j, \qquad \text{(State Variable Inequality Constraints)},$$

$$\sum_{i=1}^{m} \psi_{ij} \leq 1 \quad \text{for} \quad j = 1,\ldots,n,$$

$$\sum_{i=1}^{n} \phi_{ij} \leq 1 \quad \text{for} \quad j = 1,\ldots,m,$$

(Strategic Variable Inequality Constraints)

$$\phi_{ij}, \psi_{ij} \geq 0,$$

where

$x_i(t)$, $y_i(t)$ are force levels,

r_i, s_i are replacement rates,

v_i, w_i are the utilities assigned survivors,

a_{ij} is the rate at which one Y_j unit can destroy X_i,

b_{ij} is the rate at which one X_j unit can destroy Y_i,

ϕ_{ij} is the fraction of X_j who fire at Y_i,

and ψ_{ij} is the fraction of Y_j who fire at X_i.

The above problem (1) contains all the essential features (and can be made identical to by minor changes in formulation) of all the Lanchester-type differential games studied previously in the operations research literature (see [23]).

It should be noted that we use capital letters to denote a strategy (see [3] or [9] for precise definitions), whereas the corresponding lower case letter is used to denote the realization of the strategy (Friedman uses the term <u>outcome</u> (see pp. 22-23 of [9]). A strategy is a mapping from the state space to the space of admissible decisions. Heuristically,

it is a contingency plan that tells us what to do depending upon where we find ourselves in the state space (time may sometimes be an additional state variable). In other words, we have $\underset{\sim}{\Psi} = \underset{\sim}{\Psi}(t,\underset{\sim}{x},y)$ where $\underset{\sim}{\Psi}$ denotes a q vector,* $\underset{\sim}{x}$ denotes an m vector, and $\underset{\sim}{y}$ denotes an n vector. Then strategic variable is the realization (or outcome) of a strategy as follows:

$$\psi_{ij}(t) = \Psi_{ij}(t,\underset{\sim}{x},\underset{\sim}{y}). \tag{2}$$

As mentioned above, we will refer to (1) as a Lanchester-type differential game. This suggests that we consider the following problem (denoted as "Problem I") for the development of necessary conditions of optimality for a class of differential games with SVIC's.

Problem I.

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\quad G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{x},\underset{\sim}{y},u,v)dt ,$$

subject to:

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},v),$$

$$\dot{\underset{\sim}{y}} = \underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},u),$$

$u(t)$, $v(t)$ are unrestricted (scalar) strategic variables,

$x_m(t) \geq \alpha(t)$ for all $t \in [0,T]$ (scalar state variable inequality constraints),

$y_n(t) \leq \beta(t)$ for all $t \in [0,T]$

with <u>scalar terminal condition</u> $F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) = 0$,

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*We use standard notation for vectors, etc. as defined, for example, on pp. 4-6 of [8].

where we assume that all functions are smooth enough to insure the existence of all partial derivatives required in the following analysis. In Problem I above, $\underset{\sim}{x}$ is an m-vector of state variables, and $\underset{\sim}{y}$ is an n-vector of state variables.

3. Mathematical Preliminaries.

In this section we further establish our notation and state our assumptions. We consider closed-loop (or feedback) strategies, denoted as $U(t,\underset{\sim}{x},\underset{\sim}{y})$ and $V(t,\underset{\sim}{x},\underset{\sim}{y})$, for the players. A strategy then is the specification of what value should be given to a player's strategic variable at each point in time depending upon the time and current values of the state variables. The realization (or outcome) of a strategy will be denoted by the corresponding lower case letter. Thus

$$u(t) = U(t,\underset{\sim}{x},\underset{\sim}{y}) \quad \text{and} \quad v(t) = V(t,\underset{\sim}{x},\underset{\sim}{y}). \tag{3}$$

We call $u(t)$ and $v(t)$ strategic variables.

The criterion functional is denoted by J, and thus we have for Problem I

$$J = G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{x}(t),\underset{\sim}{y}(t),u(t),v(t))dt. \tag{4}$$

Player I chooses $U(t,\underset{\sim}{x},\underset{\sim}{y})$ and tries to maximize J, while Player II chooses $V(t,\underset{\sim}{x},\underset{\sim}{y})$ and tries to minimize J. We say that (U^*,V^*) is a saddle point of J if

$$J(u,V^*(t,\underset{\sim}{x},\underset{\sim}{y})) \le J(U^*(t,\underset{\sim}{x},\underset{\sim}{y}),V^*(t,\underset{\sim}{x},\underset{\sim}{y})) \le J(U^*(t,\underset{\sim}{x},\underset{\sim}{y}),v). \tag{SPC}$$

Then (U^*,V^*) is the solution to Problem I. We denote the value of the differential game as W. Then

$$W = J(U^*,V^*). \tag{5}$$

We assume that Problem I has value and a saddle point in pure strategies.

It will be convenient to denote the vector of state variables as

$$\underset{\sim}{z}^T = (\underset{\sim}{x}^T, \underset{\sim}{y}^T) = (x_1,\ldots,x_m,y_1,\ldots,y_n). \tag{6}$$

We will then sometimes write terms in a more convenient form as

$$L(t,\underset{\sim}{z},u,v) = L(t,\underset{\sim}{x},\underset{\sim}{y},u,v). \tag{7}$$

Although not essential for the development of necessary conditions of optimality, to insure the existence of value for the differential game we assume*

$$L(t,\underset{\sim}{z},u,v) = L_1(t,\underset{\sim}{x},\underset{\sim}{y},u) + L_2(t,\underset{\sim}{x},\underset{\sim}{y},v). \tag{8}$$

It is assumed that both players have perfect information as to the state variables. In other words, both players know precisely the current state of the system.

We assume that the first time derivative of $y_n(t)$ explicitly contains the strategic variable u and that $(\dot{y}_n)_u = g_{n_u}(t,\underset{\sim}{x},\underset{\sim}{y},u) \neq 0$ along an optimal trajectory. We further assume that the first time derivative of $x_m(t)$ explicitly contains the strategic variable v and that $(\dot{x}_m)_v = f_{m_v}(t,\underset{\sim}{x},\underset{\sim}{y},v) \neq 0$ along an optimal trajectory. In other

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*Several other technical conditions must be satisfied to guarantee the existence of value and of a saddle point in pure strategies to Problem I (see Chapter 6 of [9]).

words, we assume that $x_m(t) \geq \alpha(t)$ and $y_n(t) \leq \beta(t)$ are the appropriate types of first order SVIC's (see [12]).*

4. Statement of Necessary Conditions of Optimality for Problem I.

First we define

$$H(t,\underset{\sim}{z},\underset{\sim}{p},\underset{\sim}{q},\underset{\sim}{\eta},u,v) = L(t,\underset{\sim}{z},u,v) + \underset{\sim}{p}^T\underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},v)$$

$$+ \underset{\sim}{q}^T\underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},u) - \eta_1(t)\{x_m-\alpha(t)\} - \eta_2(t)\{y_n-\beta(t)\}, \qquad (9)$$

where

$$\eta_1(t) \begin{cases} = 0 & \text{for } x_m > \alpha(t), \\ \geq 0 & \text{for } x_m = \alpha(t), \end{cases}$$

and

$$\eta_2(t) \begin{cases} = 0 & \text{for } y_n < \beta(t), \\ \geq 0 & \text{for } y_n = \beta(t), \end{cases}$$

and

$$\Phi(T,\underset{\sim}{z}(T),\sigma) = G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \sigma F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)). \qquad (10)$$

Let $\underset{\sim}{x}^*(t)$ for $0 \leq y \leq T$ denote the optimal path which results from $(U^*(t,\underset{\sim}{x},\underset{\sim}{y}),\ V^*(t,\underset{\sim}{x},\underset{\sim}{y}))$. We assume that the solution to Problem I is normal (see [1]) and that the optimal path is not tangent to the terminal manifold or to any manifold of discontinuity of U^* or V^* or to the boundary of the state space at entry to a constrained subarc which lies on the boundary. Then, in order that the strategy pair (U^*,V^*) be a saddle point of the criterion functional it is necessary that there exist unique functions

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*In optimal control theory one says [6] that a problem has a p^{th} order SVIC when the p^{th} time derivative of the state-variable constraint is the first to explicitly contain the control variable.

$\underset{\sim}{p}(t)$, $\underset{\sim}{q}(t)$, $\eta_1(t)$, and $\eta_2(t)$ and constants σ, ν_1, and ν_2 such that the following conditions hold

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad \frac{\partial L}{\partial u} + \underset{\sim}{q}^T \frac{\partial \underset{\sim}{g}}{\partial u} = 0, \tag{11}$$

$$\frac{\partial H}{\partial v} = 0 \quad \text{or} \quad \frac{\partial L}{\partial v} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{f}}{\partial v} = 0, \tag{12}$$

$$-\dot{\underset{\sim}{p}}^T = H_{\underset{\sim}{x}} = \frac{\partial L}{\partial \underset{\sim}{x}} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} + \underset{\sim}{q}^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \eta_1 (\underset{\sim}{\Delta}^m)^T, \tag{13}$$

$$-\dot{\underset{\sim}{q}}^T = H_{\underset{\sim}{y}} = \frac{\partial L}{\partial \underset{\sim}{y}} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} + \underset{\sim}{q}^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \eta_2 (\underset{\sim}{\Delta}^n)^T, \tag{14}$$

$$\underset{\sim}{p}^T(T) = \frac{\partial \Phi}{\partial \underset{\sim}{x}(T)} - \nu_1 (\underset{\sim}{\Delta}^m)^T \quad \text{where} \quad \nu_1 \begin{cases} = 0 & \text{for} \quad x_m(T) > \alpha(T), \\ \geq 0 & \text{for} \quad x_m(T) = \alpha(T), \end{cases}{}^{*} \tag{15}$$

$$\underset{\sim}{q}^T(T) = \frac{\partial \Phi}{\partial \underset{\sim}{y}(T)} - \nu_2 (\underset{\sim}{\Delta}^n)^T \quad \text{where} \quad \nu_2 \begin{cases} = 0 & \text{for} \quad y_n(T) < \beta(T), \\ \geq 0 & \text{for} \quad y_n(T) = \beta(T), \end{cases} \tag{16}$$

$$H(T) + \frac{\partial \Phi}{\partial T} + \nu_1 \dot{\alpha}(T) + \nu_2 \dot{\beta}(T) = 0, \tag{17}$$

where

$$\underset{\sim}{\Delta}^j = \begin{pmatrix} \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{jj} \end{pmatrix} \quad \text{and} \quad \delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{otherwise.} \end{cases}$$

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*These results hold only when the boundary of the state space (at least the part for which $x_m = \alpha(t)$) is <u>non-absorbing</u> (see [24]). For an <u>absorbing</u> state boundary we have

$$\nu_1 \begin{cases} = 0 & \text{for} \quad x_m(T) > \alpha(T), \\ \geq 0 & \text{for} \quad x_m(T) = \alpha(T) \quad \text{but} \quad x_m(t) > \alpha(t) \quad \text{for} \quad t < T, \\ \underline{\text{unrestricted}} & \text{for} \quad x_m(T) = \alpha(T) \quad \text{and} \quad x_m(t) = \alpha(t) \quad \text{for} \\ & \qquad t_1 \leq t \leq T \quad \text{with} \quad t_1 < T. \end{cases}$$

Analogous to the Weierstrass condition, we also have the max-min principle

$$H(u,v^*) \le H(u^*,v^*) \le H(u^*,v), \tag{18}$$

which holds for all admissible u and v. H, $\underset{\sim}{p}(t)$, and $\underset{\sim}{q}(t)$ are continuous (vector-valued) functions except at manifolds of discontinuity of both U^* and V^*, where for all differentials dt, $d\underset{\sim}{x}$, and $d\underset{\sim}{y}$ along the manifold

$$(H^+ - H^-)dt - (\underset{\sim}{p}^+ - \underset{\sim}{p}^-)^T d\underset{\sim}{x} - (\underset{\sim}{q}^+ - \underset{\sim}{q}^-)^T d\underset{\sim}{y} = 0. \tag{19}$$

In other words, at both entrances to and exits from constrained subarcs on the boundary of the state space, we have

$$\underset{\sim}{p}(t_i^-) = \underset{\sim}{p}(t_i^+), \tag{20}$$

$$\underset{\sim}{q}(t_i^-) = \underset{\sim}{q}(t_i^+), \tag{21}$$

and

$$H(t_i^-) = H(t_i^+), \tag{22}$$

except possibly at manifolds of discontinuity of both U^* and V^*, where all the dual variables are continuous except those corresponding to x_m or y_n which describes the boundary.

The above first order necessary conditions of optimality for Problem I are analogous to the optimal control theory results of Jacobson, Lele, and Speyer [12]. The above results may also be written in a form analogous to that of optimal control theory results of Berkovitz [2] and Gamkrelidze (see Chapter VI in [14]). This is done by introducing new multipliers $\underset{\sim}{\lambda}(t)$ such that

$$\underset{\sim}{p}(t) = \underset{\sim}{\lambda}_1(t) - \mu_1(t)\underset{\sim}{\Delta}^m, \tag{23}$$

$$\underset{\sim}{q}(t) = \underset{\sim}{\lambda}_2(t) - \mu_2(t)\underset{\sim}{\Delta}^n. \tag{24}$$

It is well known (see [24]) that we may then make the following identifications

$$\eta_i(t) = -\dot{\mu}_i(t) \quad \text{for} \quad i = 1,2. \tag{25}$$

5. Development of Necessary Conditions of Optimality.

It is convenient to re-write Problem I as (DG).

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\ \{G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{x},\underset{\sim}{y},u,v)dt\},$$

$$\text{subject to:} \qquad \dot{\underset{\sim}{x}} = \underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},v), \tag{DG}$$

$$\dot{\underset{\sim}{y}} = \underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},u),$$

$u(t)$, $v(t)$ are unrestricted (scalar) strategic variables,

$x_m(t) \geq \alpha(t)$ for all $t \in [0,T]$ (scalar state variable inequality constraints),

$y_n(t) \leq \beta(t)$ for all $t \in [0,T]$

with scalar terminal condition $F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) = 0$.

Considering the saddle point condition (SPC) for the criterion functional, necessary conditions of optimality for (DG) will be developed according to the approach of Berkovitz [3] in which maximum and minimum problems associated with the game are considered. Necessary conditions of optimality are developed for each of these two problems. It is then shown that these two sets of conditions may be written as a single set of necessary conditions for (DG).

We extend Berkovitz's variational approach to differential games with SVIC's by treating boundary segments by two different methods in the associated optimal control problems. For the simple case of bounded state

variables at hand, we may think of both methods of treating an SVIC as techniques in which the dimensionality of the state space is reduced for a constrained subarc which lies on the boundary for a finite interval of time.

5.1. Necessary Conditions for Control Problem A.

Thus, we are led to maximum and minimum problems associated with (SPC). The associated maximum problem we denote as Control Problem A, (CPA). We are to determine $u^*(t) = U^*(t,\underset{\sim}{x},\underset{\sim}{y})$ such that the following holds:

$$\max_{u(t)} \{G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{z},u,V^*(t,\underset{\sim}{x},\underset{\sim}{y}))dt\},$$

$$\text{subject to:} \quad \dot{\underset{\sim}{x}} = \underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},V^*(t,\underset{\sim}{x},\underset{\sim}{y})), \tag{CPA}$$

$$\dot{\underset{\sim}{y}} = \underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},u),$$

$u(t)$ is unrestricted scalar control variable,

$x_m(t) \geq \alpha(t)$ and $y_n(t) \leq \beta(t)$ for all $t \in [0,T]$,

with scalar terminal condition $F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) = 0$.

For (CPA) we assume that the first time derivative of $y_n(t)$ explicitly contains the control variable u and that $(\dot{y}_n)_u = g_{n_u}(t,\underset{\sim}{x},\underset{\sim}{y},u) \neq 0$ along an optimal trajectory. Let us note that in (CPA) we have $(\dot{x}_m)_u \equiv 0$. Thus, variations in the control u have no direct influence on a trajectory for which $x_m(t) = \alpha(t)$. Let us recall that we have assumed that in (DG) the first time derivative of $x_m(t)$ explicitly contains the strategic variable v and that $(\dot{x}_m)_v(t,\underset{\sim}{x},\underset{\sim}{y},v) \neq 0$ along an

optimal trajectory. Thus, in (CPA) we will have (since in (CPA) $x_m(t) \geq \alpha(t)$ is a "rule of the game") $\dot{x}_m = f_m(t,\underset{\sim}{x},\underset{\sim}{y},V^*(t^+,\underset{\sim}{x},\underset{\sim}{y})) \geq \dot{\alpha}(t)$ when $x_m(t) = \alpha(t)$ where $f_m(t^+)$ denotes a right hand limit.

There are two cases to be considered in developing necessary conditions of optimality for (CPA):

(1) $x_m(t) > \alpha(t)$ almost everywhere in time,

(2) $x_m(t) = \alpha(t)$ for $t_{entry}^{X_m} \leq t \leq t_{exit}^{X_m}$ with $t_{entry}^{X_m} < t_{exit}^{X_m}$.

<u>CASE (1): $x_m(t) > \alpha(t)$ almost everywhere in time</u>.

We will apply the results of Speyer [12] (see also [17] and [24]) to (CPA). To this end, we define

$$H^A(t,\underset{\sim}{z},\underset{\sim}{p}^A,\underset{\sim}{q}^A,u) = L(t,\underset{\sim}{z},u,V^*(t,\underset{\sim}{z})) + (\underset{\sim}{p}^A)^T\underset{\sim}{f}(t,\underset{\sim}{z},V^*(t,\underset{\sim}{z}))$$
$$+ (\underset{\sim}{q}^A)^T\underset{\sim}{g}(t,\underset{\sim}{z},u) - \eta_2(t)\{y_n-\beta(t)\}, \tag{26}$$

where

$$\eta_2(t)\begin{cases} = 0 & \text{for } y_n < \beta(t), \\ \geq 0 & \text{for } y_n = \beta(t), \end{cases}$$

and

$$\Phi^A(T,\underset{\sim}{z}(T),\sigma^A) = G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \sigma^A F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)). \tag{27}$$

According to the results of Speyer [12], <u>first order necessary conditions of optimality for (CPA)</u> are as follows:

$$\frac{\partial H^A}{\partial u} = 0 \quad \text{or} \quad \frac{\partial L}{\partial u} + (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u} = 0, \tag{28}$$

$$(\dot{\underset{\sim}{p}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^A)^T\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^A)^T\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T\frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{x}}, \tag{30}$$

$$(\dot{\underset{\sim}{q}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \dot{V}^*_{\underset{\sim}{y}} + \eta_2(\underset{\sim}{\Delta}^n)^T, \quad (30)$$

$$(\underset{\sim}{p}^A(T))^T = \frac{\partial \Phi^A}{\partial \underset{\sim}{x}(T)} - \nu_1^A(\underset{\sim}{\Delta}^m)^T \quad \text{where} \quad \nu_1^A \begin{cases} = 0 & \text{for} \quad x_m(T) > \alpha(T), \\ \neq 0 & \text{for} \quad x_m(T) = \alpha(T), \end{cases} \quad (31)$$

$$(\underset{\sim}{q}^A(T))^T = \frac{\partial \Phi^A}{\partial \underset{\sim}{y}(T)} - \nu_2^A(\underset{\sim}{\Delta}^n)^T \quad \text{where} \quad \nu_2^A \begin{cases} = 0 & \text{for} \quad y_n(T) < \beta(T), \\ \geq 0 & \text{for} \quad y_n(T) = \beta(T),^* \end{cases} \quad (32)$$

$$H^A(T) + \frac{\partial \Phi^A}{\partial T} + \nu_1^A \dot{\alpha}(T) + \nu_2^A \dot{\beta}(T) = 0, \quad (33)$$

where $H^A(T)$ is an abbreviation for $H^A(t=T,\underset{\sim}{z}(T),\underset{\sim}{p}^A(T),\underset{\sim}{q}^A(T),u^*(T))$. We will make copious use of such notational conveniences. We also have the <u>maximum principle</u>

$$H^A(u,V^*(t,\underset{\sim}{z})) \leq H^A(u^*,V^*(t,\underset{\sim}{z})). \quad (34)$$

H^A, $\underset{\sim}{p}^A(t)$, and $\underset{\sim}{q}^A(t)$ are continuous except at manifolds of discontinuity of V^*, where for all differentials dt, $d\underset{\sim}{x}$, $d\underset{\sim}{y}$ along the manifold

$$(H^{A+}-H^{A-})dt - (\underset{\sim}{p}^{A+}-\underset{\sim}{p}^{A-})^T d\underset{\sim}{x} - (\underset{\sim}{q}^{A+}-\underset{\sim}{q}^{A-})^T d\underset{\sim}{y} = 0. \quad (35)$$

When $y_n(t) < \beta(t)$ almost everywhere in time, the above becomes (except at manifolds of discontinuity)

$$\frac{\partial L}{\partial u} + (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u} = 0, \quad (36)$$

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*As noted for equation (15), these results hold only when the boundary of the state space (at least the part on which $y_n = \beta(t)$) is <u>non-absorbing</u> (see [24]). The reader is referred to the previous footnote for the statement of terminal multiplier properties for an <u>absorbing</u> state boundary. Henceforth, in order to contain the length of this appendix we will leave it to the reader to supply such necessary qualifications for conditions on terminal multipliers.

$$(\dot{\underset{\sim}{p}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{x}}, \tag{37}$$

$$(\dot{\underset{\sim}{q}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{y}}. \tag{38}$$

Using (36), we may write the adjoint equations as

$$\begin{aligned}(\dot{\underset{\sim}{p}}^A)^T = &-\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (q^A)^T \frac{\partial \underset{\sim}{g}}{\partial x} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{\underset{\sim}{x}} \\ &- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{x}},\end{aligned} \tag{39}$$

$$\begin{aligned}(\dot{\underset{\sim}{q}}^A)^T = &-\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{y} \\ &- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{y}}.\end{aligned} \tag{40}$$

When $y_n(t) = \beta(t)$ for $t^{Y_n}_{entry} \le t \le t^{Y_n}_{exit}$ with $t^{Y_n}_{entry} < t^{Y_n}_{exit}$, we may reduce the dimension of the state space and also the adjoint variable space (see [17]). Let us adopt the following notation

$$\underset{\sim}{P}^T = (P_1, P_2, \ldots, P_m), \qquad \underset{\sim}{Q}^T = (Q_1, Q_2, \ldots, Q_{n-1}),$$

$$\underset{\sim}{\bar{g}}^T = (g_1, g_2, \ldots, g_{n-1}), \qquad \underset{\sim}{\bar{y}}^T = (y_1, y_2, \ldots, y_{n-1}),$$

$$\underset{\sim}{\bar{z}}^T = (\underset{\sim}{x}^T, \underset{\sim}{\bar{y}}^T) = (x_1, \ldots, x_m, y_1, \ldots, y_{n-1}),$$

$$\bar{V} = \bar{V}(t, \underset{\sim}{\bar{z}}) = V(t, \underset{\sim}{x}, \underset{\sim}{\bar{y}}, y_n = \beta(t)).$$

In other words, in the reduced state space the state variables are $x_1, \ldots, x_m, y_1, \ldots, y_{n-1}$, and the corresponding adjoint variables are denoted as $\underset{\sim}{P}^A$ and $\underset{\sim}{Q}^A$. When V^* is continuous at entry to the boundary of the state space with $y_n = \beta(t)$, the corner conditions at such junctures to this reduced state space are (for example, at entry with $t_e = t^{Y_n}_{entry}$) as follows for a first order SVIC [24]

$$\underset{\sim}{p}^A(t_e^-) = \underset{\sim}{p}^A(t_e^+), \tag{41}$$

$$\underset{\sim}{q}^A(t_e^-) = \underset{\sim}{q}^A(t_e^+), \tag{42}$$

and

$$H^A(t_e^-) = H^A(t_e^+). \tag{43}$$

Assuming that the optimal path is not tangent to the boundary of the state space at entry to the constrained subarc (i.e. $g_n(t_e^-) - \dot{\beta}(t_e) \neq 0$), it follows that

$$q_n^A(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{Q}^A(t_e^+))^T\{\underset{\sim}{\bar{g}}(t_e^+) - \underset{\sim}{\bar{g}}(t_e^-)\}}{g_n(t_e^-) - \dot{\beta}(t_e)} \tag{44}$$

where we have adopted the notation

$$\underset{\sim}{P}^A(t_e^+) = \underset{\sim}{p}^A(t_e^+),$$

and (45)

$$\underset{\sim}{Q}^A(t_e^+) = \underset{\sim}{\bar{q}}^A(t_e^+).$$

When V^* is discontinuous at entry to the boundary of the state space with $y_n = \beta(t)$, it follows from adapting the argument of Berkovitz [1] on discontinuous "right-hand sides" to the usual development of the corner conditions (see [24])

$$\underset{\sim}{p}^A(t_e^-) = \underset{\sim}{p}^A(t_e^+), \tag{46}$$

$$\underset{\sim}{q}^A(t_e^-) = \underset{\sim}{q}^A(t_e^+) + \rho\underset{\sim}{\Delta}^n, \tag{47}$$

and

$$H^A(t_e^-) = H^A(t_e^+) + \rho\dot{\beta}(t_e). \tag{48}$$

By (46) through (48) and the non-tangency assumption, it follows that

$$q_n^A(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^A(t_e^+))^T\{\underset{\sim}{f}(t_e^+) - \underset{\sim}{f}(t_e^-)\} + \underset{\sim}{Q}^A(t_e^+))^T\{\underset{\sim}{\bar{g}}(t_e^+) - \underset{\sim}{\bar{g}}(t_e^-)\}}{g_n(t_e^-) - \dot{\beta}(t_e)} \tag{49}$$

where again we have used (45). It should be noted that (49) reduces to (44) when V^* is continuous at entry. We may also re-write (46) and (47) (or (41) and (42)) as

$$\underset{\sim}{p}^A(t_e^-) = \underset{\sim}{P}^A(t_e^+),$$

and

$$\underset{\sim}{\bar{q}}^A(t_e^-) = \underset{\sim}{Q}^A(t_e^+). \tag{50}$$

The adjoint equations in the reduced state space may be obtained from (29) and (30) by use of (28) to eliminate $q_n^A(t)$. From (28), we have

$$q_n^A = -\frac{1}{\dfrac{\partial g_n}{\partial u}}\left\{\frac{\partial L}{\partial u} + (\bar{q}^A)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\},$$

since we have assumed $(g_n)_u \neq 0$ along an optimal trajectory for (CPA). Since $\underset{\sim}{\bar{q}}^A(t)$ is denoted as $\underset{\sim}{Q}^A(t)$ in the reduced state space, this becomes

$$q_n^A = -\frac{1}{\dfrac{\partial g_n}{\partial u}}\left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\}. \tag{51}$$

Using (51) and denoting the adjoint variables in the reduced state space, (29) may be written as

$$(\dot{\underset{\sim}{P}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial \underset{\sim}{x}} + \frac{\left(\dfrac{\partial g_n}{\partial \underset{\sim}{x}}\right)}{\left(\dfrac{\partial g_n}{\partial u}\right)}\left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{x}}. \tag{52}$$

Similarly for (30)

$$(\dot{\underset{\sim}{Q}}^A)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{Q}^A)^T \frac{\partial \bar{g}}{\partial \bar{\underset{\sim}{y}}} + \frac{\left(\frac{\partial g_n}{\partial \bar{\underset{\sim}{y}}}\right)}{\left(\frac{\partial g_n}{\partial u}\right)} \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial u}\right\}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{y}}} \ . \tag{53}$$

It should be noted that, to be precise, in the reduced state space we have

$$\left(\frac{\partial \bar{V}}{\partial x_k}\right)_{\substack{t,\bar{\underset{\sim}{y}} \\ x_i \text{ for } i=1,\ldots,m \\ i \neq k}} = \left(\frac{\partial V}{\partial x_k}\right)_{\substack{t,\underset{\sim}{y} \\ x_i \text{ for } i=1,\ldots,m \\ i \neq k}}$$

etc. However, to avoid excessive notation we will not be precise on this point and hope that the identification of the independent variables when partial derivatives are to be taken is clear from the context. When $y_n(t) = \beta(t)$ for $t^{Y_n}_{entry} \leq t \leq t^{Y_n}_{exit}$ with $t^{Y_n}_{entry} < t^{Y_n}_{exit}$, we have

$$\dot{y}_n = g_n(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n = \beta(t),\bar{U}^*(t,\bar{\underset{\sim}{z}})) = \dot{\beta}(t) \quad \text{for} \quad t^{Y_n}_{entry} \leq t \leq t^{Y_n}_{exit},$$

except at manifolds of discontinuity of $\bar{U}^*(t,\bar{\underset{\sim}{z}})$ (i.e. except at the end points of the interval). Differentiating the above identity with respect to $\underset{\sim}{x}$, we obtain

$$g_{m_{\underset{\sim}{x}}} + g_{n_u} \bar{U}^*_{\underset{\sim}{x}} = 0,$$

which yields

$$\bar{U}^*_{\underset{\sim}{x}} = -\frac{\left(\frac{\partial g_n}{\partial \underset{\sim}{x}}\right)}{\left(\frac{\partial g_n}{\partial u}\right)} \ , \tag{54}$$

since $\frac{\partial g_n}{\partial u} \neq 0$ along an optimal trajectory. Similarly we obtain

$$\bar{U}^*_{\bar{\underset{\sim}{y}}} = -\frac{\left(\frac{\partial g_n}{\partial \bar{\underset{\sim}{y}}}\right)}{\left(\frac{\partial g_n}{\partial u}\right)} . \tag{55}$$

Using (54) and (55), we may write (52) and (53) as

$$(\dot{\underset{\sim}{P}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{x}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{x}} , \tag{56}$$

$$(\dot{\underset{\sim}{Q}}^A)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial \bar{\underset{\sim}{y}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\bar{\underset{\sim}{y}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{y}}} . \tag{57}$$

<u>CASE (2): $x_m(t) = \alpha(t)$ for $t^{X_m}_{entry} \leq t \leq t^{X_m}_{exit}$ with $t^{X_m}_{entry} < t^{X_m}_{exit}$.</u>

We consider the optimal control problem (CPA) in a reduced state space in which the state variables are $x_1,\ldots,x_{m-1},y_1,\ldots,y_n$ (also denoted as $\bar{\underset{\sim}{z}}^T = (\bar{\underset{\sim}{x}}^T,\underset{\sim}{y}^T))$ and the corresponding adjoint variables are denoted as $\underset{\sim}{P}^A$, $\underset{\sim}{Q}^A$. Then the system dynamics are given by

$$\dot{\bar{\underset{\sim}{x}}} = \bar{\underset{\sim}{f}}(t,\bar{\underset{\sim}{x}},x_m{=}\alpha(t),\underset{\sim}{y},\bar{V}^*(t,\bar{\underset{\sim}{z}})) ,$$

$$\dot{\underset{\sim}{y}} = \underset{\sim}{g}(t,\bar{\underset{\sim}{x}},x_m{=}\alpha(t),\underset{\sim}{y},u) ,$$

with $\bar{V}^*(t,\bar{\underset{\sim}{z}}) = V^*(t,\bar{\underset{\sim}{x}},x_m{=}\alpha(t),\underset{\sim}{y})$ being determined by $f_m(t,\bar{\underset{\sim}{x}},x_m{=}\alpha(t),\underset{\sim}{y},\bar{V}^*(t,\bar{\underset{\sim}{z}})) = \dot{\alpha}(t)$. We also will denote $U(t,\bar{x},x_m{=}\alpha(t),y)$ as $\bar{U} = \bar{U}(t,\bar{z})$. In the reduced state space we define a Hamiltonian as follows

$$\bar{H}^A(t,\underset{\sim}{\bar{z}},\underset{\sim}{P}^A,\underset{\sim}{Q}^A,u) = L(t,\underset{\sim}{\bar{x}},x_m=\alpha(t),\underset{\sim}{y},u,\bar{V}^*(t,\underset{\sim}{\bar{z}}))$$

$$+ (\underset{\sim}{P}^A)^T\underset{\sim}{\bar{f}}(t,\underset{\sim}{\bar{x}},x_m=\alpha(t),\underset{\sim}{y},\bar{V}^*(t,\underset{\sim}{\bar{z}}))$$

$$+ (\underset{\sim}{Q}^A)^T\underset{\sim}{g}(t,\underset{\sim}{\bar{x}},x_m=\alpha(t),\underset{\sim}{y},u) - \bar{\eta}_2(t)\{y_n-\beta(t)\} \;, \tag{58}$$

where

$$\bar{\eta}_2(t) \begin{cases} = 0 & \text{for } \; y_n < \beta(t), \\ \geq 0 & \text{for } \; y_n = \beta(t). \end{cases}$$

According to the results of Speyer [12], <u>first order necessary conditions of optimality for (CPA) in CASE (2)</u> are as follows:

$$\frac{\partial\bar{H}^A}{\partial u} = 0 \quad \text{or} \quad \frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u} = 0, \tag{59}$$

$$(\dot{\underset{\sim}{P}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{\bar{x}}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\bar{x}}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{\bar{x}}} - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{\bar{x}}} \;, \tag{60}$$

$$(\dot{\underset{\sim}{Q}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial \underset{\sim}{y}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} \bar{V}^*_y + \bar{\eta}_2(\underset{\sim}{\Delta}^n)^T. \tag{61}$$

Using (59), we may write (60) and (61) for $y_n < \beta(t)$ as

$$(\dot{\underset{\sim}{P}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{\bar{x}}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial \underset{\sim}{\bar{x}}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{\bar{x}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{\bar{x}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{\bar{x}}} \;, \tag{62}$$

$$(\dot{\underset{\sim}{Q}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial \underset{\sim}{y}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{y}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{y}} \;. \tag{63}$$

$\bar{\underset{\sim}{H}}^A$, $\underset{\sim}{P}^A(t)$, and $\underset{\sim}{Q}^A(t)$ are continuous except at manifolds of discontinuity of V^* and possibly at junctures to the boundary of the state space with $x_m = \alpha(t)$. Thus, we also must consider necessary conditions for a juncture between an unconstrained subarc and a constrained subarc on the state boundary. The results of Speyer and Bryson [17] tell us that

$$\underset{\sim}{P}^A(t_e^+) = \bar{\underset{\sim}{p}}(t_e^-),$$

and

$$\underset{\sim}{Q}^A(t_e^+) = \underset{\sim}{q}^A(t_e^-), \tag{64}$$

where $t_e = t_{entry}^{X_m}$ and t_e^- denotes a left hand limit. It also may be shown that

$$\bar{H}^A(t_e^+) = H^A(t_e^-) - p_m^A(t_e^-)\dot{\alpha}(t_e). \tag{65}$$

This may be written as

$$L(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^+),\bar{V}^*(t_e^+,\bar{\underset{\sim}{z}})) + (\underset{\sim}{P}^A(t_e^+))^T\bar{\underset{\sim}{f}}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^+,\bar{\underset{\sim}{z}}))$$

$$+ (\underset{\sim}{Q}^A(t_e^+))^T\underset{\sim}{g}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^+)) = L(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^-),\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}}))$$

$$+ (\underset{\sim}{p}^A(t_e^-))^T\underset{\sim}{f}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}})) + (\underset{\sim}{q}^A(t_e^-))^T\underset{\sim}{g}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^-))$$

$$- p_m^A(t_e^-)\dot{\alpha}(t_e), \tag{66}$$

or (using (64))

$$p_m^A(t_e^-)\{f_m(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}})) - \dot{\alpha}(t_e)\} =$$

$$L(t_e,\underset{\sim}{x},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^+),\bar{V}^*(t_e^+,\bar{\underset{\sim}{z}})) - L(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^-),\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}}))$$

$$+ (\underset{\sim}{P}^A(t_e^+))^T\{\bar{\underset{\sim}{f}}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^+,\bar{\underset{\sim}{z}})) - \bar{\underset{\sim}{f}}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}}))\}$$

$$+ (\underset{\sim}{Q}^A(t_e^+))^T\{\underset{\sim}{g}(t_e,\underset{\sim}{x},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^+)) - \underset{\sim}{g}(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},u^*(t_e^-))\} \ . \quad (67)$$

When the state trajectory does not enter the boundary of the state space tangentially, we have that $f_m(t_e,\bar{\underset{\sim}{x}},x_m=\alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}})) - \dot{\alpha}(t_e) \neq 0$, so that (67) may be solved (uniquely) for $p_m^A(t_e^-)$. Then

$$p_m^A(t_e^-) = \frac{L(t_e^+)-L(t_e^-)+(\underset{\sim}{P}^A(t_e^+))^T\{\bar{\underset{\sim}{f}}(t_e^+)-\bar{\underset{\sim}{f}}(t_e^-)\}+(\underset{\sim}{Q}^A(t_e^+))^T\{\underset{\sim}{g}(t_e^+)-\underset{\sim}{g}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}. \quad (68)$$

When $f_m(t_e^-) - \dot{\alpha}(t_e) \neq 0$ and $u^*(t_e^-) = u^*(t_e^+)$ (i.e. no manifold of discontinuity of U^* at entry to reduced state space), (68) becomes

$$p_m^A(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^A(t_e^+))^T\{\bar{\underset{\sim}{f}}(t_e^+) - \bar{\underset{\sim}{f}}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}. \quad (69)$$

When $f_m(t_e,\bar{\underset{\sim}{x}},x_m = \alpha(t_e),\underset{\sim}{y},\bar{V}^*(t_e^-,\bar{\underset{\sim}{z}})) - \dot{\alpha}(t_e) = 0$, then the juncture conditions (64) and (65) (i.e. (67)) do not determine $p_m^A(t_e^-)$.

To summarize, we have for (CPA)

<u>for $x_m(t) > \alpha(t)$, $y_n(t) < \beta(t)$ almost everywhere in time:</u>

$$(\dot{\underset{\sim}{p}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{\underset{\sim}{x}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{x}} , \quad (70)$$

$$(\dot{\underset{\sim}{q}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{\underset{\sim}{y}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{y}} \; . \tag{71}$$

<u>for $y_n(t) = \beta(t)$ for $t^{Y_n}_{\text{entry}} \le t \le t^{Y_n}_{\text{exit}}$ with $t^{Y_n}_{\text{entry}} < t^{Y_n}_{\text{exit}}$ and $x_m(t) > \alpha(t)$ almost everywhere in time</u>:

$$(\dot{\underset{\sim}{P}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{x}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{x}} \; , \tag{72}$$

$$(\dot{\underset{\sim}{Q}}^A)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial \bar{\underset{\sim}{y}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial u}\right\} \bar{U}^*_{\bar{\underset{\sim}{y}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{y}}} \; . \tag{73}$$

<u>for $x_m(t) = \alpha(t)$ for $t^{X_m}_{\text{entry}} \le t \le t^{X_m}_{\text{exit}}$ with $t^{X_m}_{\text{entry}} < t^{X_m}_{\text{exit}}$ and $y_n(t) < \beta(t)$ almost everywhere in time</u>:

$$(\dot{\underset{\sim}{P}}^A)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{x}}} - (\underset{\sim}{P}^A)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial \bar{\underset{\sim}{x}}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \bar{\underset{\sim}{x}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\bar{\underset{\sim}{x}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{x}}} \; , \tag{74}$$

$$(\dot{\underset{\sim}{Q}}^A)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{P}^A)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial \underset{\sim}{y}} - (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^A)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{y}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^A)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{y}} \; . \tag{75}$$

5.2. Necessary Conditions for Control Problem B.

We also consider a minimum problem (denoted as Control Problem B, (CPB)) associated with (SPC). We are to determine $v^*(t) = V^*(t,\underset{\sim}{x},\underset{\sim}{y})$ such that the following holds:

$$\underset{v(t)}{\text{minimize}} \{G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{z},U^*(t,\underset{\sim}{x},\underset{\sim}{y}),v)dt\},$$

$$\text{subject to:} \qquad \dot{\underset{\sim}{x}} = \underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},v), \qquad \text{(CPB)}$$

$$\dot{\underset{\sim}{y}} = \underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},U^*(t,\underset{\sim}{x},\underset{\sim}{y})),$$

$v(t)$ is unrestricted scalar control variable,

$x_m(t) \geq \alpha(t)$ and $y_n(t) \leq \beta(t)$ for all $t \in [0,T]$,

with scalar terminal condition $F(t,\underset{\sim}{x},(T),\underset{\sim}{y}(T)) = 0$.

For (CPB) we assume that the first time derivative of $x_m(t)$ explicitly contains the control variable v and that $(\dot{x}_m)_v = f_{m_v}(t,\underset{\sim}{x},\underset{\sim}{y},v) \neq 0$ along an optimal trajectory. Let us note that in (CPB) we have $(y_n)_v \equiv 0$. Thus, variations in the control v have no direct influence on a trajectory for which $y_n(t) = \beta(t)$. Let us recall that we have assumed that in (DG) the first time derivative of $y_n(t)$ explicitly contains the strategic variable u and that $(\dot{y}_n)_u = g_{n_u}(t,\underset{\sim}{x},\underset{\sim}{y},u) \neq 0$ along an optimal trajectory. Thus, in (CPB) we will have (since in (CPB) $y_n(t) \leq \beta(t)$ is a "rule of the game") $\dot{y}_n = g_n(t,\underset{\sim}{x},\underset{\sim}{y},U^*(t^+,\underset{\sim}{x},\underset{\sim}{y})) \leq \dot{\beta}(t)$ when $y_n(t) = \beta(t)$ where $g_n(t^+)$ denotes a right hand limit.

There are two cases to be considered in developing necessary conditions of optimality for (CPB):

(1) $y_n(t) < \beta(t)$ almost everywhere in time,

(2) $y_n(t) = \beta(t)$ for $t^{Y_n}_{entry} \le t \le t^{Y_n}_{exit}$ with $t^{Y_n}_{entry} < t^{Y_n}_{exit}$.

CASE (1): $y_n(t) < \beta(t)$ almost everywhere in time.

As before, we will apply the results of Speyer to (CPB). To this end, we define

$$H^B(t,\underset{\sim}{z},\underset{\sim}{p}^B,\underset{\sim}{q}^B,v) = L(t,\underset{\sim}{z},U^*(t,\underset{\sim}{z}),v) + (\underset{\sim}{p}^B)^T\underset{\sim}{f}(t,\underset{\sim}{z},v)$$

$$+ (\underset{\sim}{q}^B)^T\underset{\sim}{g}(t,\underset{\sim}{z},U^*(t,\underset{\sim}{z})) - \eta_1(t)\{x_m-\alpha(t)\}, \qquad (76)$$

where

$$\eta_1(t) \begin{cases} = 0 & \text{for } x_m > \alpha(t), \\ \ge 0 & \text{for } x_m = \alpha(t), \end{cases}$$

and

$$\Phi^B(T,\underset{\sim}{z}(T),\sigma^B) = G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \sigma^B F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)). \qquad (77)$$

According to the results of Speyer [12], first order necessary conditions of optimality for (CPB) are as follows:

$$\frac{\partial H^B}{\partial v} = 0 \quad \text{or} \quad \frac{\partial L}{\partial v} + (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v} = 0, \qquad (78)$$

$$(\dot{\underset{\sim}{p}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{\underset{\sim}{x}} + \eta_1(\underset{\sim}{\Delta}^m)^T, \qquad (79)$$

$$(\dot{\underset{\sim}{q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{\underset{\sim}{y}}, \qquad (80)$$

$$(\underset{\sim}{p}^B(T))^T = \frac{\partial\Phi^B}{\partial \underset{\sim}{x}(T)} - \nu_1^B(\underset{\sim}{\Delta}^m)^T \quad \text{where} \quad \nu_1^B \begin{cases} = 0 & \text{for} \quad x_m(T) > \alpha(T), \\ \geq 0 & \text{for} \quad x_m(T) = \alpha(T), \end{cases} \tag{81}$$

$$(\underset{\sim}{q}^B(T))^T = \frac{\partial\Phi^B}{\partial \underset{\sim}{y}(T)} - \nu_2^B(\underset{\sim}{\Delta}^n)^T \quad \text{where} \quad \nu_2^B \begin{cases} = 0 & \text{for} \quad y_n(T) < \alpha(T), \\ \neq 0 & \text{for} \quad y_n(T) = \beta(T), \end{cases} \tag{82}$$

$$H^B(T) + \frac{\partial\Phi^B}{\partial T} + \nu_1^B\dot{\alpha}(T) + \nu_2^B\dot{\beta}(T) = 0. \tag{83}$$

We also have the minimum principle

$$H^B(U^*(t,\underset{\sim}{z}),v) \geq H^B(U^*(t,\underset{\sim}{z}),v^*). \tag{84}$$

H^B, $\underset{\sim}{p}^B(t)$, and $\underset{\sim}{q}^B(t)$ are continuous except at manifolds of discontinuity of U^*, where for all differentials dt, $d\underset{\sim}{x}$, $d\underset{\sim}{y}$ along the manifold

$$(H^{B+}-H^{B-})dt - (\underset{\sim}{p}^{B+}-\underset{\sim}{p}^{B-})^T\, d\underset{\sim}{x} - (\underset{\sim}{q}^{B+}-\underset{\sim}{q}^{B-})^T\, d\underset{\sim}{y} = 0. \tag{85}$$

When $x_m(t) > \alpha(t)$ almost everywhere in time, the above becomes (except at manifolds of discontinuity)

$$\frac{\partial L}{\partial v} + (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v} = 0, \tag{86}$$

$$(\dot{\underset{\sim}{p}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^B)^T\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^B)^T\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T\frac{\partial g}{\partial u}\}\, U^*_{\underset{\sim}{x}}, \tag{87}$$

$$(\dot{\underset{\sim}{q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^B)^T\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^B)^T\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T\frac{\partial g}{\partial u}\}\, U^*_{\underset{\sim}{y}}. \tag{88}$$

Using (86), we may write the adjoint equations as

$$(\dot{\underset{\sim}{p}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^B)^T\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^B)^T\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T\frac{\partial \underset{\sim}{g}}{\partial u}\}\, U^*_x$$
$$- \{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^B)^T\frac{\partial \underset{\sim}{f}}{\partial v}\}\, V^*_{\underset{\sim}{x}}\ , \tag{89}$$

$$(\dot{\underset{\sim}{q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U^*_{\underset{\sim}{y}}$$
$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V^*_{\underset{\sim}{y}} \,. \tag{90}$$

When $x_m(t) = \alpha(t)$ for $t^{X_m}_{entry} \le t \le r^{X_m}_{exit}$ with $t^{X_m}_{entry} < t^{X_m}_{exit}$, we may reduce the dimension of the state space and also the adjoint variable space (see [17]). Let us adopt the following notation

$$\underset{\sim}{P}^T = (P_1, P_2, \ldots, P_{m-1}), \qquad \underset{\sim}{Q}^T = (Q_1, Q_2, \ldots, Q_n),$$

$$\bar{\underset{\sim}{f}}^T = (f_1, f_2, \ldots, f_{m-1}), \qquad \bar{\underset{\sim}{x}}^T = (x_1, x_2, \ldots, x_{m-1}),$$

$$\bar{U} = \bar{U}(t, \bar{\underset{\sim}{z}}) = U(t, \bar{\underset{\sim}{x}}, x_m = \alpha(t), \underset{\sim}{y}).$$

In other words, in the reduced state space the state variables are $x_1, \ldots, x_{m-1}$, $y_1, \ldots, y_n$, and the corresponding adjoint variables are denoted as $\underset{\sim}{P}^B$, $\underset{\sim}{Q}^B$. When U^* is continuous at entry to the boundary of the state space with $x_m = \alpha(t)$, the corner conditions at such junctures to this reduced state space are (for example, at entry with $t_e = t^{X_m}_{entry}$) as follows for a first order SVIC [24]

$$\underset{\sim}{p}^B(t_e^-) = \underset{\sim}{p}^B(t_e^+), \tag{91}$$

$$\underset{\sim}{q}^B(t_e^-) = \underset{\sim}{q}^B(t_e^+), \tag{92}$$

and

$$H^B(t_e^-) = H^B(t_e^+). \tag{93}$$

Assuming that the optimal path is not tangent to the boundary of the state space at entry to the constrained subarc (i.e. $f_m(t_e^-) - \dot{\alpha}(t_e) \neq 0$), it follows that

$$p_m^B(t_e^-) = \frac{L(T_e^+) - L(t_e^-) + \underset{\sim}{P}^B(t_e^+))\{\bar{\underset{\sim}{f}}(t_e^+) - \bar{\underset{\sim}{f}}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}$$

where we have adopted the notation

$$\underset{\sim}{P}^B(t_e^+) = \underset{\sim}{\bar{p}}^B(t_e^+),$$

and (95)

$$\underset{\sim}{Q}^B(t_e^+) = \underset{\sim}{q}^B(t_e^+).$$

When U^* is discontinuous at entry to the boundary of the state space with $x_m = \alpha(t)$, it follows from adapting the argument of Berkovitz [1] on discontinuous "right-hand sides" to the usual development of the corner conditions (see [24])

$$\underset{\sim}{p}^B(t_e^-) = \underset{\sim}{p}^B(t_e^+) + \rho\underset{\sim}{\Delta}^m \tag{96}$$

$$\underset{\sim}{q}^B(t_e^-) = \underset{\sim}{q}^B(t_e^+), \tag{97}$$

and

$$H^B(t_e^-) = H^B(t_e^+) + \rho\dot{\alpha}(t_e). \tag{98}$$

By (96) through (98) and the non-tangency assumption, it follows that

$$p_m^B(t_e^-) = \frac{L(t_e^+)-L(t_e^-)+(\underset{\sim}{P}^B(t_e^+))^T\{\bar{\underset{\sim}{f}}(t_e^+)-\bar{\underset{\sim}{f}}(t_e^-)\}+(\underset{\sim}{Q}^B(t_e^+))^T\{\underset{\sim}{g}(t_e^+)-\underset{\sim}{g}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}, \tag{99}$$

where again we have used (95). It should be noted that (99) reduces to (94) when U^* is continuous at entry. We may also re-write (96) and (97) (or (91) and (92)) as

$$\underset{\sim}{\bar{p}}^B(t_e^-) = \underset{\sim}{P}^B(t_e^+),$$

and (100)

$$\underset{\sim}{q}^B(t_e^-) = \underset{\sim}{Q}^B(t_e^+).$$

The adjoint equations in the reduced state space may be obtained from (79) and (80) by use of (78) to eliminate $p_m^B(t)$. From (78), we have

$$p_m^B = -\frac{1}{\frac{\partial f_m}{\partial v}} \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{\bar{p}}^B)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} ,$$

since we have assumed $(f_m)_v \neq 0$ along an optimal trajectory for (CPB). Since $\underset{\sim}{\bar{p}}^B(t)$ is denoted as $\underset{\sim}{P}^B(t)$ in the reduced state space, this becomes

$$p_m^B = -\frac{1}{\frac{\partial f_m}{\partial v}} \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} . \tag{101}$$

Using (101) and denoting the adjoint variables in the reduced state space, (79) may be written as

$$(\dot{\underset{\sim}{P}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{\bar{x}}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial \underset{\sim}{\bar{x}}} - (Q^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{\bar{x}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{\bar{x}}}$$

$$+ \frac{\left(\frac{\partial f_m}{\partial \underset{\sim}{\bar{x}}}\right)}{\left(\frac{\partial f_m}{\partial v}\right)} \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} \tag{102}$$

Similarly for (80)

$$(\underset{\sim}{Q}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial \underset{\sim}{y}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{y}}$$

$$+ \frac{\left(\frac{\partial f_m}{\partial \underset{\sim}{y}}\right)}{\left(\frac{\partial f_m}{\partial v}\right)} \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{\bar{f}}}{\partial v}\right\} . \tag{103}$$

It should be noted that, to be precise, in the reduced state space we have

$$\left(\frac{\partial \bar{U}}{\partial x_k}\right)_{\substack{t,\underset{\sim}{y} \\ x_i \text{ for } i=1,\ldots,m-1 \\ i\neq k}} = \left(\frac{\partial U}{\partial x_k}\right)_{\substack{t,\underset{\sim}{y} \\ x_i \text{ for } i=1,\ldots,m \\ i\neq k}}$$

etc. However, to avoid excessive notation we will not be precise on this point and hope that the identification of the independent variables when partial derivatives are to be taken is clear from the context. When $x_m(t) = \alpha(t)$ for $t^{X_m}_{entry} \leq t \leq t^{X_m}_{exit}$ with $t^{X_m}_{entry} < t^{X_m}_{exit}$, we have

$$\dot{x}_m = f_m(t,\bar{\underset{\sim}{x}},x_m{=}\alpha(t),\underset{\sim}{y},\bar{V}^*(t,\bar{\underset{\sim}{z}})) = \dot{\alpha}(t) \quad \text{for} \quad t^{X_m}_{entry} \leq t \leq t^{X_m}_{exit},$$

except at manifolds of discontinuity of $\bar{V}^*(t,\bar{\underset{\sim}{z}})$ (i.e. except at the end points of the interval). Differentiating the above identity with respect to $\bar{\underset{\sim}{x}}$, we obtain

$$f_{m_{\bar{\underset{\sim}{x}}}} + f_{m_v} \bar{V}^*_{\bar{\underset{\sim}{x}}} = 0,$$

which yields

$$\bar{V}^*_{\bar{\underset{\sim}{x}}} = - \frac{\left(\dfrac{\partial f_m}{\partial \bar{\underset{\sim}{x}}}\right)}{\left(\dfrac{\partial f_m}{\partial v}\right)}, \tag{104}$$

since $\dfrac{\partial f_m}{\partial v} \neq 0$ along an optimal trajectory. Similarly we also obtain

$$\bar{V}^*_{\bar{y}} = - \frac{\left(\dfrac{\partial f_m}{\partial \underset{\sim}{y}}\right)}{\left(\dfrac{\partial f_m}{\partial v}\right)} . \tag{105}$$

Using (104) and (105), we may write (102) and (103) as

$$(\dot{\underset{\sim}{P}}^B)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{x}}} - (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial \bar{\underset{\sim}{x}}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \bar{\underset{\sim}{x}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\bar{\underset{\sim}{x}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{x}}} \quad , \tag{106}$$

$$(\dot{\underset{\sim}{Q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial \underset{\sim}{y}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{y}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{y}} \quad . \tag{107}$$

<u>CASE (2): $y_n(t) = \beta(t)$ for $t^{Y_n}_{entry} \le t \le t^{Y_n}_{exit}$ with $t^{Y_n}_{entry} < t^{Y_n}_{exit}$.</u>

We consider the optimal control problem (CPB) in a reduced state space in which the state variables are $x_1,\ldots,x_m,y_1,\ldots,y_{n-1}$ (also denoted as $\underset{\sim}{\bar{z}}^T = (\underset{\sim}{x}^T, \underset{\sim}{\bar{y}}^T))$ and the corresponding adjoint variables are denoted as $\underset{\sim}{P}^B$, $\underset{\sim}{Q}^B$. Then the system dynamics are given by

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t),v),$$

$$\dot{\bar{\underset{\sim}{y}}} = \bar{\underset{\sim}{g}}(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t),\bar{U}^*(t,\bar{\underset{\sim}{z}})),$$

with $\bar{U}^*(t,\bar{\underset{\sim}{z}}) = U^*(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t))$ being determined by $g_n(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t),$ $\bar{U}^*(t,\bar{\underset{\sim}{z}})) = \dot{\beta}(t)$. We also will denote $V(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t))$ as $\bar{V} = \bar{V}(t,\bar{\underset{\sim}{z}})$. In the reduced state space we define a Hamiltonian as follows

$$\bar{H}^B(t,\bar{\underset{\sim}{z}},\underset{\sim}{P}^B,\underset{\sim}{Q}^B,v) = L(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t),\bar{U}^*(t,\bar{\underset{\sim}{z}}),v)$$

$$+ (\underset{\sim}{P}^B)^T \underset{\sim}{f}(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t),v)$$

$$+ (\underset{\sim}{Q}^B)^T \bar{\underset{\sim}{g}}(t,\underset{\sim}{x},\bar{\underset{\sim}{y}},y_n=\beta(t),\bar{U}^*(t,\bar{\underset{\sim}{z}})) - \bar{\eta}_1(t)\{x_m-\alpha(t)\}, \tag{108}$$

where

$$\bar{\eta}_1(t) \begin{cases} = 0 & \text{for} \quad x_m > \alpha(t), \\ \geq 0 & \text{for} \quad x_m = \alpha(t). \end{cases}$$

According to the results of Speyer [12], <u>first order necessary conditions of optimality for (CPB) in CASE (2)</u> are as follows:

$$\frac{\partial \bar{H}^B}{\partial v} = 0 \quad \text{or} \quad \frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v} = 0, \tag{109}$$

$$(\dot{\underset{\sim}{P}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{x}} + \bar{\eta}_1 (\underset{\sim}{\Delta}^m)^T, \tag{110}$$

$$(\dot{\underset{\sim}{Q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{\bar{y}}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\bar{y}}} - (Q^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial \bar{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{\bar{y}}} . \tag{111}$$

Using (109), we may write (110) and (111) for $x_m > \alpha(t)$ as

$$(\dot{\underset{\sim}{P}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{x}}$$
$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{x}} , \tag{112}$$

$$(\dot{Q}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{\bar{y}}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\bar{y}}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial \underset{\sim}{\bar{y}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{\bar{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{\bar{y}}}$$
$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{\bar{y}}} . \tag{113}$$

$\bar{H}^B$, $\underset{\sim}{P}^B(t)$, and $\underset{\sim}{Q}^B(t)$ are continuous except at manifolds of discontinuity of U^* and possibly at junctures to the boundary of the state space with $y_n = \beta(t)$. Thus, we also must consider necessary conditions for a juncture between an unconstrained subarc and a constrained

subarc on the boundary. The results of Speyer and Bryson [17] tell us that

$$\underset{\sim}{P}^B(t_e^+) = \underset{\sim}{p}^B(t_e^-),$$

and (114)

$$\underset{\sim}{Q}^B(t_e^+) = \underset{\sim}{\bar{q}}^B(t_e^-),$$

where $T_e = t_{entry}^{Y_n}$ and t_e^- denotes a left hand limit. It also may be shown that

$$\bar{H}^B(t_e^+) = H^B(t_e^-) - q_n^B(t_e^-)\dot{\beta}(t_e). \tag{115}$$

This may be written as

$$L(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^+,\underset{\sim}{\bar{z}}),v^*(t_e^+)) + (\underset{\sim}{P}^B(t_e^+))^T\underset{\sim}{f}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),v^*(t_e^+))$$

$$+ (\underset{\sim}{Q}^B(t_e^+))^T\underset{\sim}{\bar{g}}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^+,\underset{\sim}{\bar{z}})) = L(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^-,\underset{\sim}{\bar{z}}),v^*(t_e^-))$$

$$+ (\underset{\sim}{p}^B(t_e^-))^T\underset{\sim}{f}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),v^*(t_e^-)) + (\underset{\sim}{q}^B(t_e^-))^T\underset{\sim}{g}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),U^*(t,\underset{\sim}{\bar{z}}))$$

$$-q_n^B(t_e^-)\dot{\beta}(t_e), \tag{116}$$

or (using (114))

$$q_n^B(t_e^-)\{g_n(t_e,x,\bar{y},y_n{=}\beta(t_e),\bar{U}^*(t_e^-,\underset{\sim}{\bar{z}})) - \dot{\beta}(t_e)\} =$$

$$L(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^+,\underset{\sim}{\bar{z}}),v^*(t_e^+)) - L(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^-,\underset{\sim}{\bar{z}}),v^*(t_e^-))$$

$$+ (\underset{\sim}{P}^B(t_e^+))^T\{\underset{\sim}{f}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),v^*(t_e^+)) - \underset{\sim}{f}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),v^*(t_e^-))\} +$$

$$(\underset{\sim}{Q}^B(t_e^+))^T\{\underset{\sim}{\bar{g}}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^+,\underset{\sim}{\bar{z}})) - \bar{g}(t_e,\underset{\sim}{x},\underset{\sim}{\bar{y}},y_n{=}\beta(t_e),\bar{U}^*(t_e^-,\underset{\sim}{\bar{z}}))\}. \tag{117}$$

When the state trajectory does not enter the boundary of the state space tangentially, we have that $g_n(t_e, \underset{\sim}{x}, \underset{\sim}{y}, y_n = \beta(t), \bar{U}^*(t_e^-, \bar{\underset{\sim}{z}})) - \dot{\beta}(t_e) \neq 0$, so that (117) may be solved (uniquely) for $q_n^B(t_e^-)$. Then

$$q_n^B(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^B(t_e^+))^T\{\underset{\sim}{f}(t_e^+) - f(t_e^-)\} + (\underset{\sim}{Q}^B(t_e^+))^T\{\bar{\underset{\sim}{g}}(t_e^+) - \bar{\underset{\sim}{g}}(t_e^-)\}}{g_n(t_e^-) - \dot{\beta}(t_e)}. \tag{118}$$

When $g_n(t_e^-) - \dot{\beta}(t_e) \neq 0$ and $v^*(t_e^+)$ (i.e. no manifold of discontinuity of V^* at entry to reduced state space), (118) becomes

$$q_n^B(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{Q}^B(t_e^+))^T\{\bar{\underset{\sim}{g}}(t_e^+) - \bar{\underset{\sim}{g}}(t_e^-)\}}{g_n(t_e^-) - \dot{\beta}(t_e)}.$$

When $g_n(t_e, \underset{\sim}{x}, \bar{\underset{\sim}{y}}, y_n = \beta(t_e), \bar{U}^*(t_e^-, \bar{\underset{\sim}{z}})) - \dot{\beta}(t_e) = 0$, then the juncture conditions (114) and (115) (i.e. (117)) do not determine $q_n^B(t_e^-)$.

To summarize, we have for (CPB)

<u>for $x_m(t) > \alpha(t)$, $y_n(t) < \beta(t)$ almost everywhere in time:</u>

$$(\dot{\underset{\sim}{p}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} - (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} = \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} U_{\underset{\sim}{x}}^* - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V_{\underset{\sim}{x}}^*, \tag{120}$$

$$(\dot{\underset{\sim}{q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - (\underset{\sim}{q}^B)^T \frac{\partial g}{\partial y} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{q}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} U_{\underset{\sim}{y}}^* - \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{p}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} V_{\underset{\sim}{y}}^*. \tag{121}$$

for $y_n(t) = \beta(t)$ for $t^{Y_n}_{entry} \le t \le t^{Y_n}_{exit}$ with $t^{Y_n}_{entry} < t^{Y_n}_{exit}$

and $x_m(t) > \alpha(t)$ almost everywhere in time:

$$(\dot{\underset{\sim}{P}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{x}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial x} - (\underset{\sim}{Q}^B)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial \underset{\sim}{x}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{x}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{x}} , \tag{122}$$

$$(\dot{\underset{\sim}{Q}}^B)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial \bar{\underset{\sim}{y}}} - (\underset{\sim}{Q}^B)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial \bar{\underset{\sim}{y}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \bar{\underset{\sim}{g}}}{\partial u}\right\} \bar{U}^*_{\bar{\underset{\sim}{y}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \underset{\sim}{f}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{y}}} . \tag{123}$$

for $x_m(t) = \alpha(t)$ for $t^{X_m}_{entry} \le t \le t^{X_m}_{exit}$ with $t^{X_m}_{entry} < t^{X_m}_{exit}$

and $y_n(t) < \beta(t)$ almost everywhere in time:

$$(\dot{\underset{\sim}{P}}^B)^T = -\frac{\partial L}{\partial \bar{\underset{\sim}{x}}} - (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial \bar{\underset{\sim}{x}}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \bar{\underset{\sim}{x}}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\bar{\underset{\sim}{x}}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial v}\right\} \bar{V}^*_{\bar{\underset{\sim}{x}}} , \tag{124}$$

$$(\dot{\underset{\sim}{Q}}^B)^T = -\frac{\partial L}{\partial \underset{\sim}{y}} - (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial \underset{\sim}{y}} - (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \left\{\frac{\partial L}{\partial u} + (\underset{\sim}{Q}^B)^T \frac{\partial \underset{\sim}{g}}{\partial u}\right\} \bar{U}^*_{\underset{\sim}{y}}$$

$$- \left\{\frac{\partial L}{\partial v} + (\underset{\sim}{P}^B)^T \frac{\partial \bar{\underset{\sim}{f}}}{\partial v}\right\} \bar{V}^*_{\underset{\sim}{y}} . \tag{125}$$

5.3. Equivalence of Multipliers in the Two Problems.

By (70) through (75) and (120) through (125), we see that $\underset{\sim}{p}^A$, $\underset{\sim}{q}^A$ and $\underset{\sim}{p}^B$, $\underset{\sim}{q}^B$ satisfy the same differential equations (appropriately modified on the boundary of the state space). In order to complete the proof of the equivalence of these multipliers it remains to show that they take on the same terminal values and possess the same continuity properties (both interior to and on the boundary of the state space).

Thus, it remains to discuss

(1) terminal values of the adjoint variables,

(2) continuity of the adjoint variables across manifolds of discontinuity of strategies at interior point of state space,

(3) juncture conditions at boundary of state space.

5.3.1. Terminal Values of the Adjoint Variables.

For (CPA), there are several distinct cases that must be considered.

CASE (A1): $x_m(T) > \alpha(T)$, $y_n(T) < \beta(T)$.

From (31), (32), and (33) we have

$$(\underset{\sim}{p}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{x}(T)} + \sigma^A \frac{\partial F}{\partial \underset{\sim}{x}(T)} ,$$

$$(\underset{\sim}{q}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{y}(T)} + \sigma^A \frac{\partial F}{\partial \underset{\sim}{y}(T)} ,$$

$$L + (\underset{\sim}{p}^A(T))^T \underset{\sim}{f} + (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \frac{\partial G}{\partial T} + \sigma^A \frac{\partial F}{\partial T} = 0.$$

We may write the above as

$$\begin{bmatrix} -\underset{\sim}{I}_m & 0 & \frac{\partial F}{\partial \underset{\sim}{x}(T)}^T \\ 0 & -\underset{\sim}{I}_n & \frac{\partial F}{\partial \underset{\sim}{y}(T)}^T \\ \underset{\sim}{f}^T & \underset{\sim}{g}^T & \frac{\partial F}{\partial T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}^A(T) \\ \underset{\sim}{q}^A(T) \\ \sigma^A \end{bmatrix} = \begin{bmatrix} -\frac{\partial G}{\partial \underset{\sim}{x}(T)}^T \\ -\frac{\partial G}{\partial \underset{\sim}{y}(T)}^T \\ -L-\frac{\partial G}{\partial T} \end{bmatrix}_{t=T}, \tag{126}$$

where $\underset{\sim}{I}_m$ denotes an $m \times m$ identity matrix. It should be recalled that we have assumed that the optimal path is not tangent to the terminal manifold. This non-tangency assumption (see [3] or [15]) yields that there is a <u>unique</u> solution to equations (126).

<u>CASE (A2): $x_m(T) = \alpha(T)$ with $x_m(t) = \alpha(t)$ for $t_{entry}^{X_m} \leq t \leq T$ with $t_{entry}^{X_m} < T$ and $y_n(T) < \beta(T)$.</u>

In this case we have

$$(\underset{\sim}{p}^A(T))^T = \frac{\partial G}{\partial \bar{\underset{\sim}{x}}(T)} + \sigma^A \frac{\partial F}{\partial \bar{\underset{\sim}{x}}(T)},$$

$$(\underset{\sim}{Q}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{y}(T)} + \bar{\sigma}^A \frac{\partial F}{\partial \underset{\sim}{y}(T)},$$

$$L + (\underset{\sim}{P}^A(T))^T \bar{\underset{\sim}{f}} + (\underset{\sim}{Q}^A(T))^T \underset{\sim}{g} + \frac{\partial G}{\partial T} + \bar{\sigma}^A \frac{\partial F}{\partial T} = 0.$$

We may write the above as

$$\begin{bmatrix} -\underset{\sim}{I}_{m-1} & 0 & \frac{\partial F}{\partial \bar{\underset{\sim}{x}}(T)}^T \\ 0 & -\underset{\sim}{I}_n & \frac{\partial F}{\partial \underset{\sim}{y}(T)}^T \\ \bar{\underset{\sim}{f}}^T & \underset{\sim}{g}^T & \frac{\partial F}{\partial T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{P}^A(T) \\ \underset{\sim}{Q}^A(T) \\ \bar{\sigma}^A \end{bmatrix} = \begin{bmatrix} -\frac{\partial G}{\partial \bar{\underset{\sim}{x}}(T)}^T \\ -\frac{\partial G}{\partial \underset{\sim}{y}(T)}^T \\ -L-\frac{\partial G}{\partial T} \end{bmatrix}_{t=T}. \tag{127}$$

Again, the non-tengency assumption yields that there is a unique solution to equations (127).

It should be pointed out that there are two ways in which we may think of equations (127) as arising:

(a) the usual transversality conditions in a reduced state space in which the state variables are $x_1,\ldots,x_{m-1},y_1,\ldots,y_n$,

(b) the usual transversality conditions in the original state space.

Let us now show that (a) and (b) are equivalent. By (31) through (33), we have in the original state space in which the state variables are $x_1,\ldots,x_m,y,\ldots,y_n$ that

$$(\underset{\sim}{p}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{x}(T)} + \sigma^A \frac{\partial F}{\partial \underset{\sim}{x}(T)} - \nu_1^A(\underset{\sim}{\Delta}^m)^T, \tag{128}$$

$$(\underset{\sim}{q}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{y}(T)} + \sigma^A \frac{\partial F}{\partial \underset{\sim}{y}(T)}, \tag{129}$$

$$H^A(T) + \frac{\partial \Phi^A}{\partial T} + \nu_1^A \dot{\alpha}(T) = 0. \tag{130}$$

We may write the last equation as

$$L + (\underset{\sim}{p}^A(T))^T \underset{\sim}{f} + (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \frac{\partial \Phi^A}{\partial T} + \nu_1^A \dot{\alpha}(T) = 0.$$

From (27) and (128) we have

$$\nu_1^A = \frac{\partial \Phi^A}{\partial x_m(T)} - p_m^A(T).$$

Thus

$$L + (\underset{\sim}{p}^A(T))^T \underset{\sim}{f} + (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \frac{\partial \Phi^A}{\partial T} + \left\{\frac{\partial \Phi^A}{\partial x_m(T)} - p_m^A(T)\right\}\dot{\alpha}(T) = 0.$$

It is convenient to write the above as

$$L + (\underset{\sim}{\bar{p}}^A(T))^T \underset{\sim}{\bar{f}} + p_m^A(T)\{f_m - \dot{\alpha}(T)\} = (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \frac{\partial \Phi^A}{\partial T} + \frac{\partial \Phi^A}{\partial x_m(T)}\,\dot{\alpha}(T) = 0. \tag{131}$$

Let us recall that $x_m = \alpha(t)$ for $t_{entry}^{X_m} \le t \le T$. Thus, we have

$$\frac{dx_m}{dt} = f_m(t,\bar{\underset{\sim}{x}},x_m=\alpha(t),\underset{\sim}{y},\bar{V}^*(t,\bar{\underset{\sim}{z}})) = \dot{\alpha}(t) \quad \text{for} \quad t_{entry}^{X_m} < t \le T,$$

or

$$f_m - \dot{\alpha}(T) = 0. \tag{132}$$

Using (132), we then obtain from (131) that

$$L + (\bar{\underset{\sim}{p}}^A(T))^T \bar{\underset{\sim}{f}} + (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \frac{\partial \Phi^A}{\partial T} + \frac{\partial \Phi^A}{\partial x_m(T)} \frac{dx_m}{dt}(T) = 0. \tag{133}$$

Recalling that the state variables in the reduced state space are $x_1,\ldots,$ $x_{m-1},y_1,\ldots,y_n$ (or $\bar{\underset{\sim}{x}}$, $\underset{\sim}{y}$), we have

$$\left(\frac{\partial \Phi^A}{\partial T}\right)_{\bar{\underset{\sim}{x}}(T),\underset{\sim}{y}(T)} = \left(\frac{\partial \Phi^A}{\partial T}\right)_{\underset{\sim}{x}(T),\underset{\sim}{y}(T)} + \left(\frac{\partial \Phi^A}{\partial x_m(T)}\right)_{T,\bar{\underset{\sim}{x}}(T),\underset{\sim}{y}(T)} \cdot \frac{dx_m}{dt}(T),$$

since on the constrained subarc we have $x_m = x_m(t)$. Hence, in the reduced state space (133) becomes

$$L + (\bar{\underset{\sim}{p}}^A(T))^T \bar{\underset{\sim}{f}} + (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \left(\frac{\partial \Phi^A}{\partial T}\right)_{\bar{\underset{\sim}{x}}(T),\underset{\sim}{y}(T)} = 0,$$

or

$$L + (\bar{\underset{\sim}{p}}^A(T))^T \bar{\underset{\sim}{f}} + (\underset{\sim}{q}^A(T))^T g + \left(\frac{\partial G}{\partial T}\right)_{\bar{\underset{\sim}{x}}(T),\underset{\sim}{y}(T)} + \sigma^A \left(\frac{\partial F}{\partial T}\right)_{\bar{\underset{\sim}{x}}(T),\underset{\sim}{y}(T)} = 0. \tag{134}$$

Making the identifications $\underset{\sim}{P}^A(T) = \bar{\underset{\sim}{p}}^A(T)$, $\underset{\sim}{Q}^A(T) = \underset{\sim}{q}^A(T)$, and $\bar{\sigma}^A = \sigma^A$, we see that (127) follows from (128), (129), and (134).

CASE (A3): $\underline{x_m(T) = \alpha(T) \text{ but } x_m(t) > \alpha(t) \text{ for } t \in (T-\delta,T) \text{ and}}$
$\underline{y_n(T) < \beta(T)}$.

In this case we essentially have an additional terminal constraint $x_m(T) = \alpha(T)$ (and the condition that $x_m(t) > \alpha(t)$ for $t \in (T-\delta,T)$ where $\delta > 0$). From (31), (32), and (33) we have

$$(\underset{\sim}{p}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{x}(T)} + \sigma^A \frac{\partial F}{\partial \underset{\sim}{x}(T)} - \nu_1^A (\underset{\sim}{\Delta}^m)^T,$$

$$(\underset{\sim}{q}^A(T))^T = \frac{\partial G}{\partial \underset{\sim}{y}(T)} + \sigma^A \frac{\partial F}{\partial \underset{\sim}{y}(T)},$$

$$L + (\underset{\sim}{p}^A(T))^T \underset{\sim}{f} + (\underset{\sim}{q}^A(T))^T \underset{\sim}{g} + \frac{\partial G}{\partial T} + \sigma^A \frac{\partial F}{\partial T} + \nu_1^A \dot{\alpha}(T) = 0.$$

We may write the above as

$$\begin{bmatrix} -\underset{\sim}{I}_m & 0 & \dfrac{\partial F}{\partial \underset{\sim}{x}(T)}^T \\ 0 & -\underset{\sim}{I}_n & \dfrac{\partial F}{\partial \underset{\sim}{y}(T)}^T \\ \underset{\sim}{f}^T & \underset{\sim}{g}^T & \dfrac{\partial F}{\partial T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}^A(T) \\ \underset{\sim}{q}^A(T) \\ \sigma^A \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial G}{\partial \underset{\sim}{x}(T)}^T + \nu_1^A (\underset{\sim}{\Delta}^m)^T \\ -\dfrac{\partial G}{\partial \underset{\sim}{y}(T)}^T \\ -L - \dfrac{\partial G}{\partial T} - \nu_1^A \dot{\alpha}(T) \end{bmatrix}_{t=T} . \tag{135}$$

The non-tangency assumption implies that equations (135) have a <u>unique</u> solution which we denote as

$$\underset{\sim}{p}^A(T) = \underset{\sim}{R}^A(\nu_1^A),$$

$$\underset{\sim}{q}^A(T) = \underset{\sim}{S}^A(\nu_1^A),$$

$$\sigma^A = \textstyle\sum^A(\nu_1^A).$$

Hence, the terminal multipliers $\underset{\sim}{p}^A(T)$, $\underset{\sim}{q}^A(T)$, σ^A depend (continuously) on a single parameter, ν_1^A. The value of this parameter, however, cannot be determined from the transversality conditions but must be determined by considerations "in the large" (see Section 7.3 of [5]). The parameter's value is chosen so that the system is steered to the given terminal state (see [25] for corresponding one-sided (optimal control) problem). Thus ν_1^A is chosen so that $x_m(T) = \alpha(T)$ but $x_m(t) > \alpha(t)$ for $t \in (T-\delta, T)$.

<u>OTHER CASES</u>. These are treated analogously to the above.

For (CPB), there are again several distinct cases that must be considered. It should be recalled (see Section 4 above) that the optimal path is not tangent to the terminal manifold.

<u>CASE (B1): $x_m(T) > \alpha(T)$, $y_n(T) < \beta(T)$</u>.

Similar to case for (CPA), we find that

$$\begin{bmatrix} -\underset{\sim}{I}_m & 0 & \dfrac{\partial F^T}{\partial \underset{\sim}{x}(T)} \\ 0 & -\underset{\sim}{I}_n & \dfrac{\partial F^T}{\partial \underset{\sim}{y}(T)} \\ \underset{\sim}{f}^T & \underset{\sim}{g}^T & \dfrac{\partial F}{\partial T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}^B(T) \\ \underset{\sim}{q}^B(T) \\ \sigma^B \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial G^T}{\partial \underset{\sim}{x}(T)} \\ -\dfrac{\partial G^T}{\partial \underset{\sim}{y}(T)} \\ -L - \dfrac{\partial G}{\partial T} \end{bmatrix}_{t=T} . \tag{136}$$

Again, the non-tangency assumption yields that (116) has a <u>unique</u> solution.

By (126) and (136) we see that $\underset{\sim}{p}^A(T)$, $\underset{\sim}{q}^A(T)$, σ^A and $\underset{\sim}{p}^B(T)$, $\underset{\sim}{q}^B(T)$, σ^B satisfy the same set of equations which have a unique solution. Hence, in this case we have shown that

$$\begin{aligned} \underset{\sim}{p}^A(T) &= \underset{\sim}{p}^B(T), \\ \underset{\sim}{q}^A(T) &= \underset{\sim}{q}^B(T), \\ \sigma^A &= \sigma^B. \end{aligned} \tag{137}$$

CASE (B2): $\underline{x_m(T) = \alpha(T) \text{ with } x_m(t) = \alpha(t) \text{ for } t_{entry}^{X_m} \le t \le T \text{ with } t_{entry}^{X_m} < T \text{ and } y_n(T) < \beta(T)}$.

Similar to case for (CPA), we find that in the reduced state space

$$\begin{bmatrix} -\underset{\sim}{I}_{m-1} & 0 & \dfrac{\partial F^T}{\partial \bar{\underset{\sim}{x}}(T)} \\ 0 & -\underset{\sim}{I}_n & \dfrac{\partial F^T}{\partial \underset{\sim}{y}(T)} \\ \bar{\underset{\sim}{f}}^T & \underset{\sim}{g}^T & \dfrac{\partial F}{\partial T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{P}^B(T) \\ \underset{\sim}{Q}^B(T) \\ \bar{\sigma}^B \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial G^T}{\partial \bar{\underset{\sim}{x}}(T)} \\ -\dfrac{\partial G^T}{\partial \underset{\sim}{y}(T)} \\ -L - \dfrac{\partial G}{\partial T} \end{bmatrix}_{t=T}. \tag{138}$$

Again, the non-tangency assumption in the reduced state space yields that (138) has a unique solution.

By (127) and (138) we see that $\underset{\sim}{P}^A(T)$, $\underset{\sim}{Q}^A(T)$, $\bar{\sigma}^A$ and $\underset{\sim}{P}^B(T)$, $\underset{\sim}{Q}^B(T)$, $\bar{\sigma}^B$ satisfy the same set of equations which have a unique solution. Hence, in this case we have shown that

$$\begin{aligned} \underset{\sim}{P}^A(T) &= \underset{\sim}{P}^B(T), \\ \underset{\sim}{Q}^A(T) &= \underset{\sim}{Q}^B(T), \\ \bar{\sigma}^A &= \bar{\sigma}^B. \end{aligned} \tag{139}$$

CASE (B3): $\underline{x_m(T) = \alpha(T) \text{ but } x_m(t) > \alpha(t) \text{ for } t \in (T-\delta, T) \text{ and } y_n(T) < \beta(T)}$.

In the necessary conditions of optimality for (CPB), we have $\nu_1^B \ge 0$. Similar to case for (CPA), we find that

$$\begin{bmatrix} -\underset{\sim}{I}_m & 0 & \dfrac{\partial F^T}{\partial \underset{\sim}{x}(T)} \\ 0 & -\underset{\sim}{I}_n & \dfrac{\partial F^T}{\partial \underset{\sim}{y}(T)} \\ \underset{\sim}{f}^T & \underset{\sim}{g}^T & \dfrac{\partial F}{\partial T} \end{bmatrix} \begin{bmatrix} \underset{\sim}{p}^B(T) \\ \underset{\sim}{q}^B(T) \\ \sigma^B \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial G^T}{\partial \underset{\sim}{x}(T)} + \nu_1^B(\underset{\sim}{\Delta}^m)^T \\ -\dfrac{\partial G^T}{\partial \underset{\sim}{y}(T)} \\ -L - \dfrac{\partial G}{\partial T} - \nu_1^B\dot{\alpha}(T) \end{bmatrix}_{t=T}. \tag{140}$$

Again, the non-tangency assumption yields that (140) has a unique solution which we denote as

$$\underset{\sim}{p}^B(T) = \underset{\sim}{R}^B(\nu_1^B),$$

$$\underset{\sim}{q}^B(t) = \underset{\sim}{S}^B(\nu_1^B),$$

$$\sigma^B = \sum{}^B(\nu_1^B).$$

As was the case for (CPA), the value of ν_1^B cannot be determined by the transversality conditions at $t = T$ but is determined by considerations "in the large."

In this third case, the problem essentially behaves as though it had two terminal equality constraints. With more than one terminal equality constraint, our results apply to differential games for which

$$\underset{\sim}{p}^A(T) = \underset{\sim}{p}^B(T),$$

$$\underset{\sim}{q}^A(T) = \underset{\sim}{q}^B(T),$$

$$\sigma^A = \sigma^B. \tag{141}$$

This type of qualification was apparently first noted by Schmitendorf and Citron [16]. We have not been able to prove (141) for differential games with $p > 1$ (where p = the number of terminal equality constraints) (nor has anybody else apparently). However, it seems plausible that (141) holds.

Let us elaborate about the plausibility of (141). In (CPA) we have $V^*(t,\underset{\sim}{x},y)$ given, and we determine $u^*(t) = U^*(t,\underset{\sim}{x},\underset{\sim}{y})$; while in (CPB) we have $U^*(t,\underset{\sim}{x},\underset{\sim}{y})$ given, and we determine $v^*(t) = V^*(t,\underset{\sim}{x},\underset{\sim}{y})$. Thus, in both problems we have the same optimal strategies $U^*(t,\underset{\sim}{x},\underset{\sim}{y})$ and $V^*(t,\underset{\sim}{x},\underset{\sim}{y})$. For a given initial point $(\underset{\sim}{x}_0^T,\underset{\sim}{y}_0^T)$ we can find the same optimal trajectory

$(\underset{\sim}{x}^*(t), \underset{\sim}{y}^*(t))$ for $0 \le t \le T$ in both problems. Hence, we would expect to be able to find the same multipliers (they may not be unique) in both problems, since we have the same adjoint equations, transversality conditions (as well as system dynamics), and optimal trajectories in both problems. $\underset{\sim}{p}^A(T)$, $\underset{\sim}{q}^A(T)$, σ^A and $\underset{\sim}{p}^B(T)$, $\underset{\sim}{q}^B(T)$, σ^B satisfy the same equations (135) and (140) (provided that we equate ν_1^A with ν_1^B). If $\underset{\sim}{p}^B(T)$, $\underset{\sim}{q}^B(T)$, σ^B is not unique, we will agree to pick it so that (141) holds.

OTHER CASES. These are treated analogously to the above.

Thus, we have shown that (137) holds in CASE (1) and (139) in CASE (2). In cases like CASE (3), our results apply to differential games in which (141) (or the equivalent) holds.

5.3.2. Continuity of the Adjoint Variables Across Manifolds of Discontinuity of Strategies at Interior Point of State Space.

Let $M(t,\underset{\sim}{x},\underset{\sim}{y}) = 0$ define the manifold of discontinuity and t_d denote the time at which this manifold is reached along a particular trajectory. Assuming that the optimal path is not tangent to any manifold of discontinuity at an interior point of the state space, standard arguments (see [3] or [15]) in which the manifold of discontinuity M plays the role of the terminal manifold then yield

$$\underset{\sim}{p}^A(t_d^-) = \underset{\sim}{p}^B(t_d^-),$$

and (142)

$$\underset{\sim}{q}^A(t_d^-) = \underset{\sim}{q}^B(_d^-).$$

Again, the non-tangency assumption yields that (140) has a unique solution which we denote as

$$\tilde{p}^B(T) = \tilde{R}^B(\nu_1^B),$$

$$\tilde{q}^B(t) = \tilde{S}^B(\nu_1^B),$$

$$\sigma^B = \textstyle\sum^B(\nu_1^B).$$

As was the case for (CPA), the value of ν_1^B cannot be determined by the transversality conditions at $t = T$ but is determined by considerations "in the large."

In this third case, the problem essentially behaves as though it had two terminal equality constraints. With more than one terminal equality constraint, our results apply to differential games for which

$$\tilde{p}^A(T) = \tilde{p}^B(T),$$

$$\tilde{q}^A(T) = \tilde{q}^B(T),$$

$$\sigma^A = \sigma^B. \tag{141}$$

This type of qualification was apparently first noted by Schmitendorf and Citron [16]. We have not been able to prove (141) for differential games with $p > 1$ (where p = the number of terminal equality constraints) (nor has anybody else apparently). However, it seems plausible that (141) holds.

Let us elaborate about the plausibility of (141). In (CPA) we have $V^*(t,\tilde{x},y)$ given, and we determine $u^*(t) = U^*(t,\tilde{x},\tilde{y})$; while in (CPB) we have $U^*(t,\tilde{x},\tilde{y})$ given, and we determine $v^*(t) = V^*(t,\tilde{x},\tilde{y})$. Thus, in both problems we have the same optimal strategies $U^*(t,\tilde{x},\tilde{y})$ and $V^*(t,\tilde{x},\tilde{y})$. For a given initial point $(\tilde{x}_0^T,\tilde{y}_0^T)$ we can find the same optimal trajectory

$(\underset{\sim}{x}^*(t),\underset{\sim}{y}^*(t))$ for $0 \le t \le T$ in both problems. Hence, we would expect to be able to find the same multipliers (they may not be unique) in both problems, since we have the same adjoint equations, transversality conditions (as well as system dynamics), and optimal trajectories in both problems. $\underset{\sim}{p}^A(T)$, $\underset{\sim}{q}^A(T)$, σ^A and $\underset{\sim}{p}^B(T)$, $\underset{\sim}{q}^B(T)$, σ^B satisfy the same equations (135) and (140) (provided that we equate ν_1^A with ν_1^B). If $\underset{\sim}{p}^B(T)$, $\underset{\sim}{q}^B(T)$, σ^B is not unique, we will agree to pick it so that (141) holds.

<u>OTHER CASES</u>. These are treated analogously to the above.

Thus, we have shown that (137) holds in CASE (1) and (139) in CASE (2). In cases like CASE (3), our results apply to differential games in which (141) (or the equivalent) holds.

5.3.2. <u>Continuity of the Adjoint Variables Across Manifolds of Discontinuity of Strategies at Interior Point of State Space</u>.

Let $M(t,\underset{\sim}{x},\underset{\sim}{y}) = 0$ define the manifold of discontinuity and t_d denote the time at which this manifold is reached along a particular trajectory. Assuming that the optimal path is not tangent to any manifold of discontinuity at an interior point of the state space, standard arguments (see [3] or [15]) in which the manifold of discontinuity M plays the role of the terminal manifold then yield

$$\underset{\sim}{p}^A(t_d^-) = \underset{\sim}{p}^B(t_d^-),$$

and (142)

$$\underset{\sim}{q}^A(t_d^-) = \underset{\sim}{q}^B(_d^-).$$

If M is a manifold of discontinuity for only U^*, then in (CPA) the usual corner conditions at an interior point of the state space yield

$$\underset{\sim}{p}^A(t_d^-) = \underset{\sim}{p}^A(t_d^+),$$

and (143)

$$\underset{\sim}{q}^A(t_d^-) = \underset{\sim}{q}^A(t_d^+).$$

Similarly, if M is a manifold of discontinuity for only V^*, then in (CPB) the usual corner conditions at an interior point of the state space yield

$$\underset{\sim}{p}^B(t_d^-) = \underset{\sim}{p}^B(t_d^+),$$

and (144)

$$\underset{\sim}{q}^B(t_d^-) = \underset{\sim}{q}^B(t_d^+).$$

If M is a manifold of discontinuity of both U^* and V^*, then we have, for example, in (CPA) that [1] for all differentials dt_d, $d\underset{\sim}{x}$, $d\underset{\sim}{y}$ along the manifold

$$\{H^A(t_d^+)-H^A(t_d^-)\}dt_d - \{\underset{\sim}{p}^A(t_d^+)-\underset{\sim}{p}^A(t_d^-)\}^T d\underset{\sim}{x} - \{\underset{\sim}{q}^A(t_d^+)-\underset{\sim}{q}^A(t_d^-)\}^T d\underset{\sim}{y} = 0. \quad (145)$$

5.3.3. <u>Juncture Conditions at Boundary of State Space.</u>

Let us consider the juncture conditions which must hold between the adjoint variables at the entrance to a constrained subarc on which $x_m(t) = \alpha(t)$ for $t \in [t_{entry}^{X_m}, t_{exit}^{X_m}]$ with $t_{entry}^{X_m} < t_{exit}^{X_m}$. Other cases are similarly treated. We denote $t_{entry}^{X_m}$ as t_e.

For (CPA) we have shown that

$$\bar{\underset{\sim}{p}}^A(t_e^-) = \underset{\sim}{P}^A(t_e^+), \quad (146)$$

$$\underset{\sim}{q}^A(t_e^-) = \underset{\sim}{Q}^A(t_e^+), \quad (147)$$

and

$$H^A(t_e^-) - p_m^A(t_e^-)\dot{\alpha}(t_e) = \bar{H}^A(t_e^+). \tag{148}$$

Assuming that the optimal trajectory does not enter the boundary of the state space tangentially, (146), (147), and (148) yield a unique value for $p_m^A(t_e^-)$.

$$p_m^A(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^A(t_e^+))^T\{\underset{\sim}{\bar{f}}(t_e^+) - \underset{\sim}{\bar{f}}(t_e^-)\} + (\underset{\sim}{Q}^A(t_e^+))^T\{\underset{\sim}{g}(t_e^+) - \underset{\sim}{g}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}.$$

When there is no manifold of discontinuity of U* at entry to the reduced state space, the above further simplifies to

$$p_m^A(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^A(t_e^+))^T\{\underset{\sim}{\bar{f}}(t_e^+) - \underset{\sim}{\bar{f}}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}. \tag{150}$$

For (CPB) we have shown that (for a first order SVIC) assuming that the optimal trajectory is not tangent to the boundary of the state space at a juncture point

$$\underset{\sim}{\bar{p}}^B(t_e^-) = \underset{\sim}{P}^B(t_e^+), \tag{151}$$

$$p_m^B(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^B(t_e^+))^T\{\underset{\sim}{\bar{f}}(t_e^+)-\underset{\sim}{\bar{f}}(t_e^+)\} + (\underset{\sim}{Q}^B(t_e^+))^T\{\underset{\sim}{g}(t_e^+)-\underset{\sim}{g}(t_e^-)\}}{f_m(t_e^-) - \dot{\alpha}(t_e)}, \tag{152}$$

and

$$\underset{\sim}{q}^B(t_e^-) = \underset{\sim}{Q}^B(t_e^+). \tag{153}$$

When there is no manifold of discontinuity of U* at entry to the reduced state space, equation (152) further simplifies to

$$p_m^B(t_e^-) = \frac{L(t_e^+) - L(t_e^-) + (\underset{\sim}{P}^B(t_e^+))^T \; \bar{\underset{\sim}{f}}(t_e^+) - \bar{\underset{\sim}{f}}(t_e^-)}{f_m(t_e^-) - \dot{\alpha}(t_e)} . \tag{154}$$

In the original state space we have (recall that we denote $\bar{\underset{\sim}{p}}^B(t_e^+)$ as $\underset{\sim}{p}^B(t_e^+)$, etc., in the reduced state space) when U^* is continuous at entry

$$\underset{\sim}{p}^B(t_e^-) = \underset{\sim}{p}^B(t_e^+),$$

and (155)

$$\underset{\sim}{q}^B(t_e^-) = \underset{\sim}{q}^B(t_e^+).$$

The above yield that when $\underset{\sim}{p}^A(t) = \underset{\sim}{p}^B(t)$ and $\underset{\sim}{q}^A(t) = \underset{\sim}{q}^B(t)$ for $t_e < t \leq T$, it follows that provided the optimal trajectory is not tangent to the boundary of the state space at a juncture point

$$\underset{\sim}{p}^A(t_e^-) = \underset{\sim}{p}^B(t_e^-),$$

and (156)

$$\underset{\sim}{q}^A(t_e^-) = \underset{\sim}{q}^B(t_e^-).$$

5.3.4. <u>Equivalence of the Adjoint Variables $\underset{\sim}{\lambda}^A(t)$ and $\underset{\sim}{\lambda}^B(t)$.</u>

It is convenient to denote

$$(\underset{\sim}{\lambda}^A(t))^T = ((\underset{\sim}{p}^A(t))^T, (\underset{\sim}{q}^A(t))^T), \tag{157}$$

and similarly for $\underset{\sim}{\lambda}^B(t)$. The equivalence of $\underset{\sim}{\lambda}^A(t)$ and $\underset{\sim}{\lambda}^B(t)$ follows from the facts that

(1) on the terminal manifold we have $\underset{\sim}{\lambda}^A(T) = \underset{\sim}{\lambda}^B(T)$,

(2) as we work backwards from the terminal manifold to the first manifold of discontinuity, both $\lambda^A(t)$ and $\lambda^B(t)$ satisfy the same differential equation so that $\underset{\sim}{\lambda}^A(t) = \underset{\sim}{\lambda}^B(t)$ for $t_1 < t \leq T$ where t_1 denotes the time closest to T such that either U^* or V^* is discontinuous,

(3) working backwards across the first manifold of discontinuity,* we have $\underset{\sim}{\lambda}^A(t_1^-) = \underset{\sim}{\lambda}^B(t_1^-)$,

(4) in this manner one works backwards across each manifold of discontinuity and shows that $\underset{\sim}{\lambda}^A(t) = \underset{\sim}{\lambda}^B(t)$ for $0 \le t \le T$.

Thus, (70) through (75), (120) through (125), (37), (139), (141), (142), and (156) yield that

$$\text{for } 0 \le t \le T: \qquad \underset{\sim}{p}^A(t) = \underset{\sim}{p}^B(t), \quad \text{and} \quad \underset{\sim}{q}^A(t) = \underset{\sim}{q}^B(t). \tag{158}$$

6. A Problem with More General Dynamics.

The above development also applies to the following problem (denoted as "Problem II") with slightly more general system dynamics.

Problem II.

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}} \left\{G(T,w(T),\underset{\sim}{x}(T),y(T)) + \int_0^T L(t,w,\underset{\sim}{x},y,u,v)dt\right\},$$

$$\text{subject to:} \qquad \dot{w} = f(t,w,\underset{\sim}{x},y,v),$$

$$\dot{\underset{\sim}{x}} = \underset{\sim}{g}(t,w,\underset{\sim}{x},y,u,v),$$

$$\dot{y} = h(t,w,\underset{\sim}{x},y,u),$$

$u(t)$, $v(t)$ are unrestricted (scalar) strategic variables,

$w(t) \ge \alpha(t)$ for all $t \in [0,T]$ (scalar state variable inequality constraints),

$y(t) \le \beta(t)$ for all $t \in [0,T]$

with <u>scalar terminal condition</u> $F(T,w(T),\underset{\sim}{x}(T),y(T)) = 0$,

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*Considering the results of Sections 5.3.2 and 5.3.3, we see that the subsequent statement applies whether the manifold of discontinuity lies interior to the state space or on the boundary.

where we assume that all functions are smooth enough to insure the existence of all partial derivatives required. In Problem II above, $\underset{\sim}{x}$ is an n-vector of state variables.

It will be convenient to denote the vector of state variables as

$$\underset{\sim}{z}^T = (w,\underset{\sim}{x}^T,y) = (w,x_1,x_2,\ldots,x_n,y).$$

We will then sometimes write terms in a more convenient form as

$$L(t,\underset{\sim}{z},u,v) = L(t,w,\underset{\sim}{x},y,u,v).$$

Before we state necessary conditions of optimality for Problem II,* let us state the <u>assumptions</u> that we make for Problem II:

(1) the differential game has value and a saddle point in pure strategies,

(2) the solution to Problem II is <u>normal</u> (see [1]),

(3) the optimal path is not tangent to the terminal manifold or to any manifold of discontinuity of U^* or V^* or to the boundary of the state space at entry to a constrained subarc which lies on the boundary,

(4) to insure the existence of value for the differential game we assume**

 (a) $L(t,\underset{\sim}{z},y,v) = L_1(t,\underset{\sim}{z},u) + L_2(t,\underset{\sim}{z},v)$,

 (b) $\underset{\sim}{g}(t,\underset{\sim}{z},u,v) = \underset{\sim}{g}_1(t,\underset{\sim}{z},u) + \underset{\sim}{g}_2(t,\underset{\sim}{z},v)$,

(5) the first time derivative of $y(t)$ explicitly contains the strategic variable u and that $(\dot{y})_u = h_u(t,w,\underset{\sim}{x},y,u) \neq 0$ along an optimal trajectory; we further assume that $(\dot{y})_v \equiv 0$, i.e. h does not depend on v at all,

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*The development of these necessary conditions is similar to that for Problem I.

**As noted above, several other technical conditions must be satisfied to guarantee the existence of value and of a saddle point in pure strategies to Problem II (see Chapter 6 of [9]).

(6) the first time derivative of $w(t)$ explicitly contains the strategic variable v and that $(\dot{w})_v = f_v(t,w,\underset{\sim}{x},y,v) \neq 0$ along an optimal trajectory; we further assume that $(\dot{w})_u \equiv 0$, i.e. f does not depend on u at all.*

Now we define

$$H(t,\underset{\sim}{z},\underset{\sim}{P},\underset{\sim}{\eta},u,v) = L(t,\underset{\sim}{z},u,v) + \lambda f(t,\underset{\sim}{z},v) + \underset{\sim}{p}^T \underset{\sim}{g}(t,\underset{\sim}{z},u,v)$$

$$+ qh(t,\underset{\sim}{z},u) - \eta_1(t)\{w-\alpha(t)\} - \eta_2(t)\{y-\beta(t)\}, \tag{159}$$

where

$$\eta_1(t) \begin{cases} = 0 & \text{for } w < \alpha(t), \\ \geq 0 & \text{for } w = \alpha(t), \end{cases}$$

and

$$\eta_2(t) \begin{cases} = 0 & \text{for } y < \beta(t), \\ \geq 0 & \text{for } y = \beta(t), \end{cases}$$

and

$$\Phi(T,z(T),\sigma) = G(T,w(T),\underset{\sim}{x}(T),y(T)) + \sigma F(T,w(T),\underset{\sim}{x}(T),y(T)). \tag{160}$$

In order that the strategy pair (U^*,V^*) be a saddle point of the criterion functional it is necessary that there exist unique functions $\lambda(t)$, $\underset{\sim}{p}(t)$, $q(t)$, $\eta_1(t)$, and $\eta_2(t)$ and constants σ, ν_1, and ν_2 such that the following conditions hold

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad \frac{\partial L}{\partial u} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{g}}{\partial u} + q \frac{\partial h}{\partial u} = 0, \tag{161}$$

$$\frac{\partial H}{\partial v} = 0 \quad \text{or} \quad \frac{\partial L}{\partial v} + \lambda \frac{\partial f}{\partial v} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{g}}{\partial v} = 0, \tag{162}$$

$$-\dot{\lambda} = H_w = \frac{\partial L}{\partial w} + \lambda \frac{\partial f}{\partial w} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{g}}{\partial w} + q \frac{\partial h}{\partial w} - \eta_1, \tag{163}$$

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*In other words, we assume that $w(t) \geq \alpha(t)$ and $y(t) \leq \beta(t)$ are the appropriate types of first order SVIC's.

$$-\dot{\underset{\sim}{p}}^T = H_{\underset{\sim}{x}} = \frac{\partial L}{\partial \underset{\sim}{x}} + \lambda \frac{\partial f}{\partial \underset{\sim}{x}} + p^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} + q \frac{\partial h}{\partial \underset{\sim}{x}}, \tag{164}$$

$$-\dot{q} = H_y = \frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{g}}{\partial y} + q \frac{\partial h}{\partial y} - \eta_2, \tag{165}$$

$$\lambda(T) = \frac{\partial \Phi}{\partial w(T)} - \nu_1 \quad \text{where} \quad \nu_1 \begin{cases} = 0 & \text{for } w(T) > \alpha(T), \\ \geq 0 & \text{for } w(T) = \alpha(T),^* \end{cases} \tag{166}$$

$$\underset{\sim}{p}^T(T) = \frac{\partial \Phi}{\partial \underset{\sim}{x}(T)}, \tag{167}$$

$$q(T) = \frac{\partial \Phi}{\partial y(T)} - \nu_2 \quad \text{where} \quad \nu_2 \begin{cases} = 0 & \text{for } y(T) < \beta(T), \\ \geq 0 & \text{for } y(T) = \beta(T), \end{cases} \tag{168}$$

$$H(T) + \frac{\partial \Phi}{\partial T} - \nu_1 \dot{\alpha}(T) - \nu_2 \dot{\beta}(T) = 0. \tag{169}$$

Analogous to the Weierstrass condition, we also have the max-min principle

$$H(u,v^*) \leq H(u^*,v^*) \leq H(u^*,v), \tag{170}$$

which holds for all admissible u and v. H, $\lambda(t)$, $\underset{\sim}{p}(t)$, and $q(t)$ are continuous functions except at manifolds of discontinuity of both U^* and V^*, where for all differentials dt, dw, $d\underset{\sim}{x}$, dy along the manifold

$$(H^+ - H^-)dt - (\lambda^+ - \lambda^-)dw - (\underset{\sim}{p}^+ - \underset{\sim}{p}^-)^T d\underset{\sim}{x} - (q^+ - q^-)dy = 0. \tag{171}$$

In other words, at both entrances to and exits from constrained subarcs on the boundary of the state space, we have

$$\lambda(t_e^-) = \lambda(t_e^+), \quad \underset{\sim}{p}(t_e^-) = \underset{\sim}{p}(t_e^+), \quad q(t_e^-) = q(t_e^+), \tag{172}$$

and

$$H(t_e^-) = H(t_e^+), \tag{173}$$

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*It should be recalled that this statement must be modified for an absorbing state boundary.

except possibly at manifolds of discontinuity of both U^* and V^*, where all the dual variables are continuous except those corresponding to w or y which describes the boundary.

The above first order necessary conditions of optimality for Problem II are analogous to the optimal control theory results of Jacobson, Lele, and Speyer [12]. These results may also be written in a form analogous to that of optimal control theory results of Berkovitz [2] and Gamkrelidze (see Chapter VI in [14]). This is done by introducing new multipliers $\underset{\sim}{\lambda}(t)$ such that

$$\lambda(t) = \lambda_1(t) - \mu_1(t), \tag{174}$$

$$\underset{\sim}{p}(t) = \underset{\sim}{\lambda}_2(t), \tag{175}$$

$$q(t) = \lambda_3(t) - \mu_2(t). \tag{176}$$

It is well known that we may then make the following identifications

$$\eta_i(t) = -\dot{\mu}_i(t) \quad \text{for} \quad i = 1,2. \tag{177}$$

7. <u>A More General Type of State Variable Inequality Constraint</u>.

It is of considerable interest and import to consider a more general type of SVIC. We, therefore, consider the following problem (denoted as "Problem III").

Problem III.

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\ \{G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{x},\underset{\sim}{y},u,v)dt\},$$

subject to: $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},v)$,

$\dot{\underset{\sim}{y}} = \underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},u)$,

$u(t)$, $v(t)$ are unrestricted (scalar) strategic variables,

$C_1(t,\underset{\sim}{x}) \geq 0$ for all $t \in [0,T]$ (scalar state variable inequality constraints),

$C_2(t,\underset{\sim}{y}) \leq 0$ for all $t \in [0,T]$

with scalar terminal condition $F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) = 0$,

where we assume that all functions are smooth enough to insure the existence of all partial derivatives required. In Problem III above, $\underset{\sim}{x}$ is an m-vector of state variables, and $\underset{\sim}{y}$ is an n-vector of state variables. It will again be convenient to adopt the notational conventions (6) and (7).

We again make the assumptions (1) through (3) given in Section 6 above and modify (4) through (6) as follows:

$(\hat{4})$ to insure the existence of value for the differential game we assume $L(t,\underset{\sim}{z},u,v) = L_1(t,\underset{\sim}{z},u) + L_2(t,\underset{\sim}{z},v)$,

$(\hat{5})$ the first time derivative of $C_2(t,\underset{\sim}{y}(t))$ explicitly contains the strategic variable u and that $(\dot{C}_2)_u(t,\underset{\sim}{x},\underset{\sim}{y},u(t)) \neq 0$ along an optimal trajectory; we further assume that $(\dot{C}_2)_v = \frac{\partial}{\partial v}\left(\frac{\partial C_2}{\partial t} + \frac{\partial C_2}{\partial \underset{\sim}{y}}\underset{\sim}{g}\right) \equiv 0$, i.e. $\dot{C}_2$ does not depend on v at all.

$(\hat{6})$ the first time derivative of $C_1(t,\underset{\sim}{x}(t))$ explicitly contains the strategic variable v and that $(\dot{C}_1)_v(t,\underset{\sim}{x},\underset{\sim}{y},v(t)) \neq 0$ along an optimal trajectory; we further assume that $(\dot{C}_1)_u = \frac{\partial}{\partial u}\left(\frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial \underset{\sim}{x}}\underset{\sim}{f}\right) \equiv 0$, i.e. $\dot{C}_1$ does not depend on u at all.

Unfortunately, it is not convenient to develop necessary conditions for Problem III in the same manner in which they were developed for Problem I. We will now state necessary conditions of optimality for Problem III. Although a proof has been developed for these conditions, it will be given elsewhere in the future, since it is rather lengthy.

First we define

$$H(t,\underset{\sim}{z},\underset{\sim}{p},\underset{\sim}{q},\underset{\sim}{\eta},u,v) = L(t,\underset{\sim}{z},u,v) + \underset{\sim}{p}^T\underset{\sim}{f}(t,\underset{\sim}{x},\underset{\sim}{y},v)$$

$$+ \underset{\sim}{q}^T\underset{\sim}{g}(t,\underset{\sim}{x},\underset{\sim}{y},u) - \eta_1(t)C_1(t,\underset{\sim}{x}) - \eta_2(t)C_2(t,\underset{\sim}{y}), \tag{178}$$

where for $i = 1,2$

$$\eta_i(t) \begin{cases} = 0 & \text{for} \quad (-1)^i C_i \le 0, \\ \ge 0 & \text{for} \quad C_i = 0, \end{cases}$$

and

$$\Phi(T,\underset{\sim}{z}(T),\sigma) = G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \sigma F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)). \tag{179}$$

In order that the strategy pair (U^*,V^*) be a saddle point of the criterion functional it is necessary that there exist unique functions $\underset{\sim}{p}(t)$, $\underset{\sim}{q}(t)$, $\eta_1(t)$, and $\eta_2(t)$ and constants ν_1, ν_2, and σ such that the following conditions hold

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad \frac{\partial L}{\partial u} + \underset{\sim}{q}^T \frac{\partial \underset{\sim}{g}}{\partial u} = 0, \tag{180}$$

$$\frac{\partial H}{\partial v} = 0 \quad \text{or} \quad \frac{\partial L}{\partial v} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{f}}{\partial v} = 0, \tag{181}$$

$$-\dot{\underset{\sim}{p}}^T = H_{\underset{\sim}{x}} = \frac{\partial L}{\partial \underset{\sim}{x}} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} + \underset{\sim}{q}^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} - \eta_1 \frac{\partial C_1}{\partial \underset{\sim}{x}}, \tag{182}$$

$$-\dot{\underset{\sim}{q}}^T = H_{\underset{\sim}{y}} = \frac{\partial L}{\partial \underset{\sim}{y}} + \underset{\sim}{p}^T \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{y}} + \underset{\sim}{q}^T \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{y}} - \eta_2 \frac{\partial C_2}{\partial \underset{\sim}{y}}, \tag{183}$$

$$\underset{\sim}{p}^T(T) = \frac{\partial\Phi}{\partial \underset{\sim}{x}(T)} - \nu_1 \frac{\partial C_1}{\partial \underset{\sim}{x}(T)} \quad \text{where} \quad \nu_1 \begin{cases} = 0 & \text{for} \quad C_1(T,\underset{\sim}{x}(T)) > 0, \\ \geq 0 & \text{for} \quad C_1(T,\underset{\sim}{x}(T)) = 0,^* \end{cases} \tag{184}$$

$$\underset{\sim}{q}^T(T) = \frac{\partial\Phi}{\partial \underset{\sim}{y}(T)} - \nu_2 \frac{\partial C_2}{\partial \underset{\sim}{y}(T)} \quad \text{where} \quad \nu_2 \begin{cases} = 0 & \text{for} \quad C_2(T,\underset{\sim}{y}(T)) < 0, \\ \geq 0 & \text{for} \quad C_2(T,\underset{\sim}{y}(T)) = 0, \end{cases} \tag{185}$$

$$H(T) + \frac{\partial\Phi}{\partial T} + \nu_1 \frac{\partial C_1}{\partial T} + \nu_2 \frac{\partial C_2}{\partial T} = 0. \tag{186}$$

Analogous to the Weierstrass condition, we also have the max-min principle (18). H, $\underset{\sim}{p}(t)$, and $\underset{\sim}{q}(t)$ are continuous functions except at manifolds of discontinuity of both U^* and V^*, where for all differentials dt, $d\underset{\sim}{x}$, $d\underset{\sim}{y}$ along the manifold (19) holds.

The above first order necessary conditions of optimality for Problem III are analogous to the optimal control theory results of Jacobson, Lele, and Speyer. These results may also be written in a form analogous to that of optimal control theory results of Berkovitz and Gamkrelidze. This is done by introducing new multipliers $\underset{\sim}{\lambda}(t)$ such that

$$\begin{aligned} \underset{\sim}{p}^T(t) &= \underset{\sim}{\lambda}_1^T(t) - \mu_1(t) \frac{\partial C_1}{\partial \underset{\sim}{x}} , \\ \underset{\sim}{q}^T(t) &= \underset{\sim}{\lambda}_2^T(t) - \mu_2(t) \frac{\partial C_2}{\partial \underset{\sim}{y}} . \end{aligned} \tag{187}$$

It is well known that we may then make the identifications (25).

As should be clear to the reader, Problem I is a special case of Problem III. However, the development of necessary conditions of optimality for Problem III is much more tedious and not the same as for Problem I. The development of these more general conditions will be given in the future.

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*It should again be recalled that this statement must be modified for an absorbing state boundary.

8. <u>Further Extensions</u>.

There are two directions of import in which the above results may be extended:

(a) additional inequality constraints on the strategies,

(b) higher order SVIC's.

It is of interest to consider a problem (denoted as "Problem IV") similar to Problem or Problem II in which there are additional inequality constraints on the (vector) strategies $\underset{\sim}{U}$ and $\underset{\sim}{V}$.

Problem IV.

$$\underset{\underset{\sim}{U}}{\text{maximize}}\ \underset{\underset{\sim}{V}}{\text{minimize}}\ \{G(T,w(T),\underset{\sim}{x}(T),y(T)) + \int_0^T L(t,w,\underset{\sim}{x},y,\underset{\sim}{u},\underset{\sim}{v})dt\},$$

subject to:

$$\dot{w} = f(t,w,\underset{\sim}{x},y,\underset{\sim}{v}),$$

$$\dot{\underset{\sim}{x}} = \underset{\sim}{g}(t,w,\underset{\sim}{x},y,\underset{\sim}{u},\underset{\sim}{v}),$$

$$\dot{y} = h(t,w,\underset{\sim}{x},y,\underset{\sim}{u}),$$

$\underset{\sim}{u}(t)$ is p-vector of strategic variables which satisfies
$\underset{\sim}{u}(t) \in \Omega^u(t,w,\underset{\sim}{x},y)$ for all $t \in [0,T]$,

$\underset{\sim}{v}(t)$ is q-vector of strategi variables which satisfies
$\underset{\sim}{v}(t) \in \Omega^v(t,w,\underset{\sim}{x},y)$ for all $t \in [0,T]$,

$w(t) \geq \alpha(t)$ for all $t \in [0,T]$
$y(t) \leq \beta(t)$ for all $t \in [0,T]$
(scalar state variable inequality constraints),

with <u>scalar terminal condition</u> $F(T,w(T),\underset{\sim}{x}(T),y(T)) = 0$,

where

$$\Omega^u = \{\underset{\sim}{u} \mid \underset{\sim}{K}(t,w,\underset{\sim}{x},y,\underset{\sim}{u}) \leq 0\},$$

and

$$\Omega^V = \{\underset{\sim}{v} \mid \underset{\sim}{R}(t,w,\underset{\sim}{x},y,\underset{\sim}{v}) \geq 0\}.$$

It is a straightforward matter to combine the previous development of Berkovitz [3] with our development for Problem I to develop necessary conditions of optimality for Problem IV. Our applications of the results of this appendix will be to Lanchester-type differential games of this form (see equations (1) above).

Another extension of import is the extension of our results on differential games with SVIC's to problems with higher order SVIC's. Similar one-sided optimization problems have arisen in our study of Lanchester-type optimal control problems [26] (see also [23]). Therefore, we consider the following problem (denoted as "Problem V").

Problem V.

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\ \{G(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) + \int_0^T L(t,\underset{\sim}{x},\underset{\sim}{y},u,v)dt\},$$

$$\text{subject to:}\quad \dot{\underset{\sim}{x}}_1 = \underset{\sim}{f}_1(t,\underset{\sim}{x}_1,\underset{\sim}{x}_2),$$

$$\dot{\underset{\sim}{x}}_2 = \underset{\sim}{f}_2(t,\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{y}_1,\underset{\sim}{y}_2,v),$$

$$\dot{\underset{\sim}{y}}_1 = \underset{\sim}{g}_1(t,\underset{\sim}{y}_1,\underset{\sim}{y}_2),$$

$$\underset{\sim}{y}_2 = \underset{\sim}{g}_2(t,\underset{\sim}{x}_1,\underset{\sim}{x}_2,\underset{\sim}{y}_1,\underset{\sim}{y}_2,u),$$

$u(t)$, $v(t)$ are unrestricted (scalar) strategic variables,

$C_1(t,\underset{\sim}{x}_1) \geq 0$ for all $t \in [0,T]$ (scalar state variable

$C_2(t,\underset{\sim}{y}_1) \leq 0$ for all $t \in [0,T]$ inequality constraints),

with <u>scalar terminal condition</u> $F(T,\underset{\sim}{x}(T),\underset{\sim}{y}(T)) = 0$,

where

$$\underset{\sim}{x}^T = (\underset{\sim}{x}_1^T,\underset{\sim}{x}_2^T) \quad\text{and}\quad \underset{\sim}{y}^T = (\underset{\sim}{y}_1^T,\underset{\sim}{y}_2^T).$$

We make the following assumptions about $C_1(t,\underset{\sim}{x}_1)$ in Problem V:

(1) $\overset{k}{(C_1)}_v \equiv 0$ for $k = 1,\ldots,p-1$ where $\overset{k}{(C_1)} = \frac{d^k C}{dt^k}$,

(2) the first derivative of $C_1(t,x_1(t))$ which explicitly contains the strategic variable v in the $p^{\underline{th}}$ and $\overset{p}{(C_1)}_v \neq 0$ along an optimal trajectory,

(3) $\overset{k}{(C_1)}_u \equiv 0$ for $k = 1,2,\ldots$ (i.e. for all integers $k \geq 1$).

Similar assumptions are made about $C_2(t,\underset{\sim}{y}_1)$. Other assumptions that are required in the development of necessary conditions are similar to those made in previous sections. The results of Jacobson, Lele, and Speyer [12] are key to establishing first order necessary conditions of optimality for Problem V.

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APPENDIX B: A Differential Game Model for the Study of Optimal Air-War Strategies.

1. Introduction.

An important question for defense planners is what are appropriate missions over the course of a tactical campaign for tactical air power. The answer to this question has far reaching implications for Navy air forces (both carrier-based and land-based) (and, of course, the Air Force). The quest for an answer to this type of question has been in large part the genesis of modern quantitative methods of military analysis. One has only to recall the classical work of Lanchester [10] or that reported in Morse and Kimball [11] (see especially pp. 73-77). More recently, the USAF Studies and Analysis Group has been using quantitative methodology [24] in trying to answer such questions.

There are many analysis approaches to answering the question of the "best"role (over time) of tactical air support. These range from operational gaming (see [12] or [23] for discussion of terminology and background) to analytical solution of an idealized differential game. However, as pointed out by L. Berkovitz and M. Dresher (see p. 612 of [1]), "operational gaming is not a helpful device for solving a game or getting significant information about the solution." We believe that there is much to be gained by a more analytical study of an idealization of the basic problem (see also [18] or [19]). Therefore, we will consider a simplified differential game model for the "best" (in the sense discussed in Section below) use of air power, with our eye on Navy applications.

As we have noted in Appendix A and elsewhere [18], [19], and important gap in the previous theory of differential games has been the lack

of multiplier conditions for differential games with state variable inequality constraints (SVIC's). It should be noted (see also [19], [21]) that state variable inequality constraints are present in all Lanchester-type dynamic tactical allocation problems because force levels (which are represented by state variables) are required to be non-negative (or some equivalent condition). Thus, the theoretical state-of-the-art previously was not adequate to solve all such problems. Hence, this (our study of the below problem) is an important application of our new theoretical results given in Appendix A. It should be pointed out that the current state-of-the-art in optimal control theory (at least its application in operations research) has not even been sufficient to allow routine solution of Lanchester-type optimal control problems (see [19]) because of the presence of SVIC's (see [17], [22]).

In this appendix then we will consider an idealization of a tactical air-war game. Versions of this problem have been considered by a number of workers [1], [2], [5], [6], and [7]. A particularly thorough discussion of modelling aspects of the problem is to be found in [6]. This is the best reference on modelling aspects that the author has encountered so far and is the source of many ideas. A special version of the problem that we will consider has been studied by R. Isaacs (see pp. 96-104 in [7]). Although he obtains a correct and thorough solution to the problem he considered, Isaacs does not employ the equivalent of optimality conditions for differential games with SVIC's. The optimality of constrained subarcs is not examined, and Isaacs' approach may not yield optimal strategies for more general problems.

A. Friedman (see pp. 239-240 of [4]) also considers Isaacs' "War of Attrition and Attack." His treatment is not nearly as thorough as that of Isaacs, and Friedman's solution is not complete. He (as does Sternberg [16]) misses certain subtle aspects first observed by Isaacs (see pp. 102-104 of [7] and also pp. 948-949 of [9]). Moreover, the emphasis of Friedman's research [4] is on the existence of value for differential games with SVIC's. The analogues of the well-known control theory multiplier conditions for constrained subarcs [8] are not developed. In fact, the treatment of the example on pp. 239-240 of [4] by Friedman is inadequate, since his approach fails to yield optimal strategies for minor modifications in the problem's formulation (see below). Thus, in this appendix we present a different and more general analysis than has appeared in the literature previously.

The problem that we study in this appendix is quite significant from the standpoint of military operations analysis. Currently, the USAF is using a similar (although much more detailed) model (TAC CONTENDER) [24], which has been widely used at the DOD/JCS planning level. Our research should shed light on cause-effect relationships between modelling assumptions and optimal air-war strategies obtained from TAC CONTENDER.

One may consider that there are four essential parts of any dynamical combat optimization problem:

(1) the decision criteria (for both combatants),

(2) the model of conflict termination conditions (and/or unit breakpoints),

(3) the model of combat dynamics,

(4) the decision latitude of the combatants.

By decision latitude we mean the extent of choices that decision makers on both sides can actually control. The other aspects have been discussed in [19]. Let us mention here that recently Pugh and Mayberry [15] have discussed the first topic (measures of effectiveness) (see also [14]). They also propose a computational scheme [14] (Lagrange dynamic programming [13]). Our analytical work here (and in the future) provides test cases for the adequacy of their computational approach.

Our analysis approach is more idealized than the TAC CONTENDER model or the conceptual approach of Pugh and Mayberry [14]. However, we feel that our research complements these other efforts. We view that our differential game analysis has several significant uses:

(1) provide guidance for higher-resolution studies,

(2) identify cause-effect relationships between optimal tactics and modelling assumptions.

2. <u>The Model</u>: <u>A Generalization of the Tactical Air-War Game</u>.

We consider an idealized model in order to gain insights into optimal air-war tactics. Background material on this model can be found in [6] (or [1]). The problem that we consider is a generalization of the tactical air-war game studied by R. Isaacs [7] (i.e. his "War of Attrition and Attack"). The origin of this problem is apparently due to A. Mengel (see [6]).

Consider a war between X and Y forces. A land war takes place and its location will be denoted as the FEBA (<u>F</u>orward <u>E</u>dge of the <u>B</u>attle <u>A</u>rea). Both the X and Y forces have a single type of aircraft that can fly two types of missions: (1) <u>ground support</u> which means flying fire support missions for the ground forces to influence the outcome of the land

war and (2) counter-air which results in the shooting down of enemy planes (but no direct help for the ground forces). The problem for each commander is to find the "best" use of his air power ($x(t)$ denotes the number of X aircraft). The decision making criterion for each commander is the net amount of support for the FEBA measured in terms of the "value" of total missions flown (or, equivalently, weighted tons of ordnance dropped or weighted total ground support-aircraft days). For planning purposes, the air war lasts for a prescribed duration of time, denoted as T, and residual values of surviving aircraft (based on linear utility for both sides' surviving aircraft) is considered.* It is assumed that new aircraft are introduced on both sides at constant rates. This situation is shown diagrammatically in Figure 1.

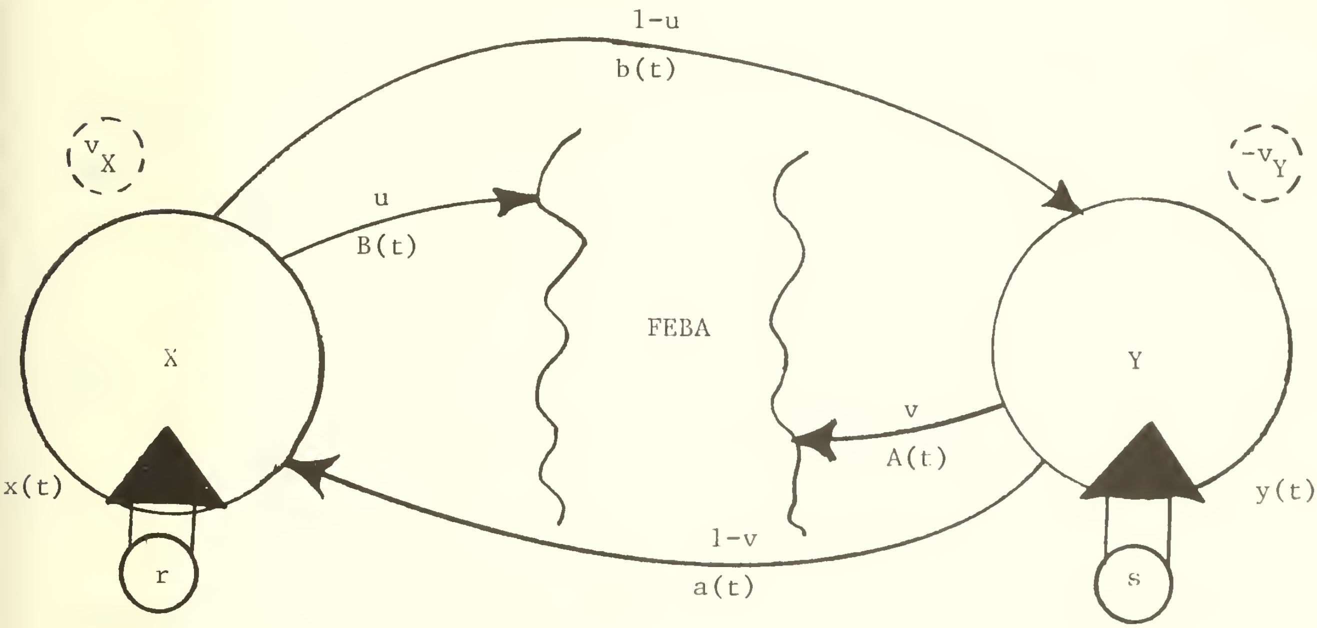

Figure 1. Diagram of Generalization of Tactical Air-War Game.

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*I am indebted to MAJ Ron Kronz, USAF for pointing out that this is commonly done in similar USAF analyses (i.e. TAC CONTENDER).

Mathematically the problem may be stated as

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\ \{v_X x(t_f) - v_Y y(t_f) + \int_0^{t_f} [B(t)ux - A(t)vy]dt\},$$

with stopping rule: $t_f - T = 0$,

subject to:
(air-battle dynamics)

$$\frac{dx}{dt} = r - (1-v)a(t)y, \qquad (1)$$

$$\frac{dy}{dt} = s - (1-u)b(t)x,$$

with initial conditions $x(t=0) = x_0$ and $y(t=0) = y_0$,

and

$x,y \geq 0$ (State Variable Inequality Constraints),

$0 \leq u,v \leq 1$ (Strategic Variable Inequality Constraints),

where

$x(t)$ and $y(t)$ are the numbers of X and Y aircraft, respectively, at time t,

$a(t)$ and $b(t)$ are variable (Lanchester) attrition-rate coefficients,

r and s are replacement rates,

$v_X(v_Y)$ is the value of one surviving $X(Y)$ aircraft,

$A(t)$ $(B(t))$ is the value of one $Y(X)$ aircraft flying ground support for a unit of time at time t,

$u(v)$ is the fraction of total $X(Y)$ aircraft which fly ground support missions.

A discussion of the circumstances under which the above air-war equations have been hypothesized to apply can be found in [20]. It should be noted that capital letters are used to denote (closed-loop) strategies, i.e. $U = U(t,x,y)$ and $V = V(t,x,y)$, whereas lower case letters are used to denote the strategic variables which are their outcomes (or realizations

of the strategies), i.e. $u = u(t) = U(t,x,y)$. Optimal (and also extremal†) strategies will be denoted as follows: U^* and V^*.

3. Development of Basic Necessary Conditions of Optimality.

It should be clear that we have in (1) above that $v_X, v_Y \geq 0$, that $r,s > 0$, and that $a(t), b(t), A(t), B(t) > 0$ for $t \in [0,T]$. Friedman has shown that the differential game (1) has value (see pp. 231-232 of [4]), and it may be shown that a saddle point exists in pure strategies (see pp. 234-235 of [4]). We now will develop necessary conditions of optimality for (1).

Applying the results of Appendix A, we have that the Hamiltonian is given by

$$H(t,x,y,p,q,\mu,\eta,u,v) = B(t)xu - A(t)yv +$$

$$p\{r-(1-v)a(t)y\} + q\{s-(1-u)b(t)x\} - \mu(t)x + \eta(t)y, \qquad (2)$$

where

$$\mu(t) \begin{cases} = 0 & \text{for } x > 0, \\ \geq 0 & \text{for } x = 0, \end{cases}$$

and

$$\eta(t) \begin{cases} = 0 & \text{for } y > 0, \\ \geq 0 & \text{for } y = 0. \end{cases}$$

We have adopted above the following correspondence between state and adjoint variables:

state variable	dual variable
x	p,
y	q.

†We use the term extremal to denote a trajectory on which the necessary conditions of optimality (see Appendix A) are satisfied at every point in time.

It is also convenient to define

$$\Phi(t_f, x(t_f), y(t_f), \sigma) = v_X x(t_f) - v_Y y(t_f) - \sigma(t_f - T). \tag{3}$$

The adjoint system of differential equations for the dual variables is

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -B(t)u^* + (1-u^*)b(t)q + \mu, \tag{4}$$

$$\frac{dq}{dt} = -\frac{\partial H}{\partial y} = A(t)v^* + (1-v^*)a(t)p - \eta. \tag{5}$$

The boundary conditions at $t = t_f$ for the adjoint variables are

$$p(t=t_f) = v_X - \nu_1 \quad \text{where} \quad \nu_1 \begin{cases} = 0 & \text{for} \quad x(t_f) > 0, \\ \geq 0 & \text{for} \quad x(t_f) = 0, \end{cases} \tag{6}$$

$$q(t=t_f) = -v_Y + \nu_2 \quad \text{where} \quad \nu_2 \begin{cases} = 0 & \text{for} \quad y(t_f) > 0, \\ \geq 0 & \text{for} \quad y(t_f) = 0. \end{cases} \tag{7}$$

We also have the transversality condition

$$H(t_f) + \frac{\partial \Phi}{\partial t_f} = 0,$$

which yields that $\sigma = H(t_f)$. This latter condition, however, is not useful for determining optimal strategies.

When $x, y > 0$, the extremal strategic-variable pair, denoted as (u^*, v^*), is determined by the max-min principle. Hence, we consider

$$\underset{0 \leq u \leq 1}{\text{maximize}} \; \underset{0 \leq v \leq 1}{\text{minimize}} \; H(t, x, y, p, q, \mu=0, \eta=0, u, v), \tag{8}$$

which by (2) yields

$$\underset{0 \leq u \leq 1}{\text{maximize}} \; xu\{B(t) + b(t)q\}, \tag{9}$$

and

$$\underset{0\le v\le 1}{\text{minimize}}\ y(-v)\{A(t) - a(t)p\}. \tag{10}$$

Thus for $x,y > 0$, we have

$$u^*(t) = \begin{cases} 1 & \text{for} \quad (-q(t)) < \dfrac{B(t)}{b(t)}, \\ 0 & \text{for} \quad (-q(t)) > \dfrac{B(t)}{b(t)}, \end{cases} \tag{11}$$

and

$$v^*(t) = \begin{cases} 1 & \text{for} \quad p(t) < \dfrac{A(t)}{a(t)}, \\ 0 & \text{for} \quad p(t) > \dfrac{A(t)}{a(t)}. \end{cases} \tag{12}$$

We must further investigate the possibility of singular subarcs (see Chapter 8 of [3]) on which, for example, $\frac{\partial H}{\partial u} = 0$ for a finite interval of time (so that all its time derivatives vanish). For a u-singular subarc, the condition that $\frac{\partial H}{\partial u} = 0$ with $x,y > 0$ yields that on a u-singular subarc we must have

$$q(t) = -\frac{B(t)}{b(t)}. \tag{13}$$

The condition $\frac{d}{dt}\left(\frac{\partial H}{\partial u}\right) = 0$ along with (12) then yields

$$A(t)v^* + (1-v^*)a(t)p(t) = -\frac{d}{dt}\left\{\frac{B(t)}{b(t)}\right\}, \tag{14}$$

or, equivalently, by (12)

$$\text{maximum}\ \{A(t),a(t)p(t)\} = -\frac{d}{dt}\left\{\frac{B(t)}{b(t)}\right\}. \tag{15}$$

Considering the adjoint equations, it is unlikely that (15) would hold for a finite interval of time. Moreover, when $a(t)$, $b(t)$, $A(t)$, and $B(t)$ are all constants (e.g. $a(t) = \bar{a} = \text{constant}$), it is immediately seen that (15) cannot hold, since

$$\text{maximum }\{\bar{A}, \bar{a}p(t)\} \geq \bar{A} > 0 = -\frac{d}{dt}\left\{\frac{\bar{B}}{\bar{b}}\right\}.$$

Thus, it is impossible (in general) to have a u-singular subarc of an optimal trajectory for (1). In a similar fashion, it is readily shown that it is impossible (in general) to have a v-singular subarc.

3.1. <u>Necessary Conditions of Optimality on Constrained Subarc for x.</u>

On a constrained subarc on which $x(t) = 0$ for $t_e^X \leq t \leq t_\ell^X$ the Y strategic variable $v^*(t)$ is determined by $\frac{dx}{dt} = 0$ for $t_e^X < t < t_\ell^X$ and hence

$$(1-v^*) = \frac{r}{a(t)y(t)},$$

or

$$v^*(t) = 1 - \frac{r}{a(t)y(t)} \quad \text{for} \quad t_e^X < t < t_\ell^X. \tag{16}$$

Since we must have $0 \leq v^* \leq 1$ and $\frac{dx}{dt}(t_e^{X-}) < 0$ (here t_e^{X-} denotes a left-hand limit), such a constrained subarc can occur only when

$$r \leq a(t)y(t) \quad \text{for} \quad t_e^X < t \leq t_\ell^X, \tag{17}$$

with $r < a(t_e^X)y(t_e^X)$. Since $x(t) = 0$ for $t_e^X \leq t \leq t_\ell^X$, the dimension of the state space may be thought of as being reduced by one, and hence the dimension of the adjoint variable space is reduced by one through $\frac{\partial H}{\partial v} = 0$. This yields

$$p(t) = \frac{A(t)}{a(t)} \quad \text{for} \quad t_e^X \leq t \leq t_\ell^X. \tag{18}$$

The multiplier $\mu(t)$ is determined by $\frac{d}{dt}\left(\frac{\partial H}{\partial v}\right) = 0$ and hence

$$\mu(t) = B(t)u^* + (1-u^*)b(t)(-q(t)) + \frac{d}{dt}\left\{\frac{A(t)}{a(t)}\right\} \quad \text{for} \quad t_e^X < t < t_\ell^X. \tag{19}$$

Thus, the necessary condition of optimality $\mu(t) \geq 0$ leads to that on a constrained subarc with $x(t) = 0$ for a finite interval of time we must have

$$B(t)u^* + (1-u^*)b(t)(-q(t)) \geq -\frac{d}{dt}\left\{\frac{A(t)}{a(t)}\right\} \quad \text{for} \quad t_e^X < t < t_\ell^X. \qquad (20)$$

When $y > 0$ and $x(t) = 0$ for $t_e^X \leq t \leq t_\ell^X$, we have that $v^*(t)$ is determined by the condition that $\frac{dx}{dt} \equiv 0$. It should be noted that (18) implies that (10) is identically satisfied. In this case only u^* is to be determined by the <u>max-min principle</u> (8). However, when $x(t) = 0$, then the max-min principle (or, equivalently, (9)) no longer yields that we <u>must</u> have $u^*(t)$ given by (11). However, we may still take (11) to hold by a continuity argument in which we consider $x(t) = \epsilon > 0$ and then let $\epsilon \to 0$. Thus, (11) may be used to write (20) as

$$\text{maximum } \{B(t), b(t)(-q(t))\} \geq -\frac{d}{dt}\left\{\frac{A(t)}{a(t)}\right\} \quad \text{for} \quad t_e^X < t < t_\ell^X. \qquad (21)$$

<u>Remark</u>: The significance of (20) is that the optimality of Y pursuing a strategy which results in $x(t) = 0$ for a finite interval of time depends upon what X's optimal strategy for using the aircraft would be if he had them. This type of behavior in which the optimality of one player steering the system along a state-constrained subarc depends upon the other player's optimal strategy is unique to differential games. Furthermore, it apparently has not been noted in the literature previously.

It remains to discuss the <u>corner conditions</u> which must hold at entrances to and exits from constrained subarcs on which $x(t) = 0$ for a finite interval of time. <u>At entrance</u> to constrained subarc with $x(t) = 0$ for $t_e^X \leq t \leq t_\ell^X$ (see Figure 2 below), we must have

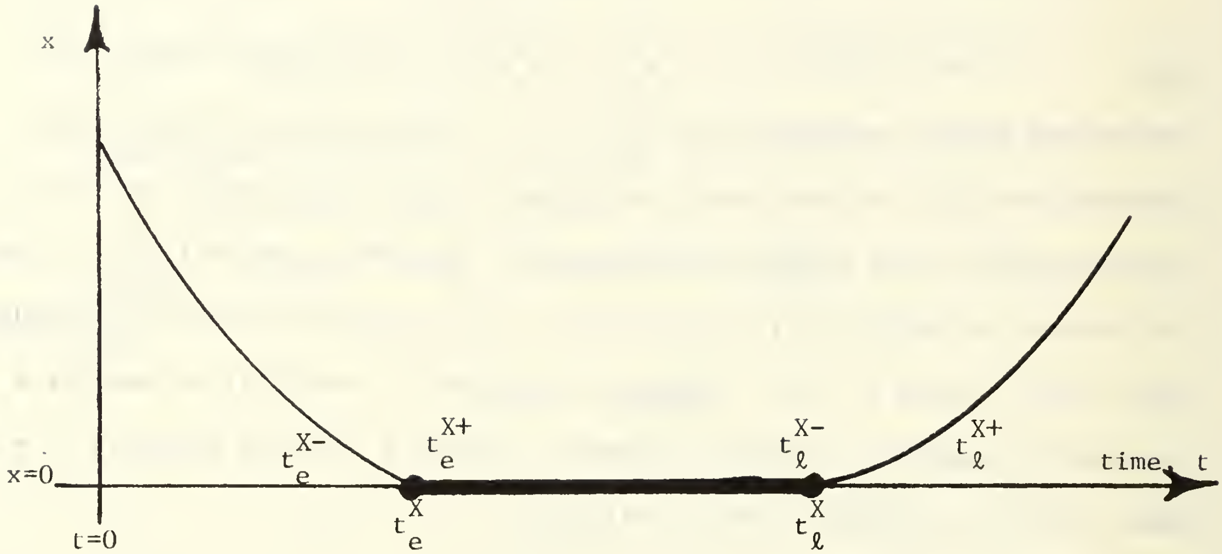

Figure 2. Entry to and Exit from Constrained Subarc.

$$p(t_e^{X-}) = p(t_e^{X+}) = p(t_e^X),$$
$$q(t_e^{X-}) = q(t_e^{X+}) = q(t_e^X), \tag{22}$$
$$H(t_e^{X-}) = H(t_e^{X+}),$$

where $H(t_e^{X-})$ denotes $H(t_e^X, x(t_e^X)=0, y(t_e^X), p(t_e^{X-}), q(t_e^{X-}), \mu=0, \eta=0, u^*(t_e^{X-}), v^*(t_e^{X-})=1)$. It is readily seen that

$$H(t_e^{X-}) = p(t_e^X)\{r - a(t_e^X)y(t_e^X)\} + q(t_e^X)s,$$

and that

$$H(t_e^{X+}) = -A(t_e^X)y(t_e^X)\{1 - \frac{r}{a(t_e^X)y(t_e^X)}\} + q(t_e^X)s,$$

so that $H(t_e^{X-}) = H(t_e^{X+}) \Rightarrow p(t_e^X)\{r - a(t_e^X)y(t_e^X)\} = \frac{A(t_e^X)}{a(t_e^X)}\{r - a(t_e^X)y(t_e^X)\}$,

whence

$$p(t_e^X) = \frac{A(t_e^X)}{a(t_e^X)}, \tag{23}$$

since we must have $\frac{dx}{dt}(t_e^{X-}) < 0$. We thus see that the corner conditions (22) are automatically satisfied when (16) through (20) hold, and thus they yield no new information.

<u>At exit</u> from constrained subarc with $x(t) = 0$ for $t_e^X \le t \le t_\ell^X$, we must have

$$\begin{aligned} p(t_\ell^{X-}) &= p(t_\ell^{X+}) = p(t_\ell^X), \\ q(t_\ell^{X-}) &= q(t_\ell^{X+}) = q(t_\ell^X), \\ H(t_\ell^{X-}) &= H(t_\ell^{X+}). \end{aligned} \tag{24}$$

It is readily seen that

$$H(t_\ell^{X-}) = -A(t_\ell^X)y(t_\ell^X)\{1 - \frac{r}{a(t_\ell^X)y(t_\ell^X)}\} + q(t_\ell^X)s,$$

and that

$$H(t_\ell^{X+}) = -A(t_\ell^X)y(t_\ell^X) + (t_\ell^X)r + q(t_\ell^X)s,$$

so that $H(t_\ell^{X-}) = H(t_\ell^{X+}) \Rightarrow$

$$p(t_\ell^X) = \frac{A(t_\ell^X)}{a(t_\ell^X)}. \tag{25}$$

Again, we see that the corner conditions (24) are automatically satisfied when (16) through (20) hold, and thus they yield no new information.

3.2. Necessary Conditions of Optimality on Constrained Subarc for y.

Similar analysis yields for a constrained subarc on which $y(t) = 0$ for $t_e^Y \le t \le t_\ell^Y$ that we must have

$$u^*(t) = 1 - \frac{s}{b(t)y(t)} \quad \text{for} \quad t_e^Y < t < t_\ell^Y, \tag{26}$$

with $s \le b(t)y(t)$ for $t_e^Y < t \le t_\ell^Y$ and $s < b(t_e^Y)y(t_e^Y)$. Additionally, we must have

$$(-q(t)) = \frac{B(t)}{b(t)} \quad \text{for} \quad t_e^Y \le t \le t_\ell^Y, \tag{27}$$

and

$$\eta(t) = A(t)v^* + (1-v^*)a(t)p(t) + \frac{d}{dt}\left\{\frac{B(t)}{b(t)}\right\} \ge 0 \quad \text{for} \quad t_e^Y < t < t_\ell^Y. \tag{28}$$

The latter condition may also be written as

$$A(t)v^* + (1-v^*)a(t)p(t) \ge -\frac{d}{dt}\left\{\frac{B(t)}{b(t)}\right\} \quad \text{for} \quad t_e^Y < t < t_\ell^Y, \tag{29}$$

or by use of (12) (which was developed from (10) for $y > 0$)

$$\text{maximum } \{A(t), a(t)p(t)\} \ge -\frac{d}{dt}\left\{\frac{B(t)}{b(t)}\right\} \quad \text{for} \quad t_e^Y < t < t_\ell^Y. \tag{30}$$

The corner conditions yield that at entry to constrained subarc on which $y(t) = 0$ for $t_e^Y \le t \le t_\ell^Y$, we must have

$$\begin{aligned} p(t_e^{Y-}) &= p(t_e^{Y+}) = p(t_e^Y), \\ q(t_e^{Y-}) &= q(t_e^{Y+}) = q(t_e^Y), \\ (-q(t_e^Y)) &= \frac{B(t_e^Y)}{b(t_e^Y)} . \end{aligned} \tag{31}$$

Also, the corner conditions yield that at exit from constrained subarc on which $y(t) = 0$ for $t_e^Y \le t \le t_\ell^Y$, we must have

$$p(t_\ell^{Y-}) = p(t_\ell^{Y+}) = p(t_\ell^{Y}),$$

$$q(t_\ell^{Y-}) = q(t_\ell^{Y+}) = q(t_\ell^{Y}), \tag{32}$$

$$(-q(t_\ell^{Y})) = \frac{B(t_\ell^{Y})}{b(t_\ell^{Y})}.$$

3.3. Discussion of Optimality Conditions for Constrained Subarcs.

In this section we will discuss some implications of the above necessary conditions of optimality that must hold on constrained subarcs. Let us first consider the case in which the following hold:

$$a(t) = \bar{a} = \text{constant},$$

$$b(t) = \bar{b} = \text{constant},$$

$$A(t) = \bar{A} = \text{constant},$$

$$B(t) = \bar{B} = \text{constant}.$$

In this case, the necessary condition of optimality (21) which must hold on constrained subarcs on which $x(t) = 0$ for $t_e^X \le t \le t_\ell^X$ with $t_e^X < t_\ell^X$ is automatically satisfied, since we have

$$\text{maximum } \{\bar{B}, \bar{b}(-q(t))\} \ge \bar{B} > 0 = -\frac{d}{dt}\{\frac{\bar{A}}{\bar{a}}\}.$$

Similarly, (29) (or equivalently (30)) is automatically satisfied on constrained subarcs on which $y(t) = 0$ for $t_e^Y \le t \le t_\ell^Y$ with $t_e^Y < t_\ell^Y$. Thus, we see that necessary conditions of optimality* are always satisfied for the holding of enemy aircraft at a zero level for a finite interval of time in the case of constant coefficients (i.e. the functions $a(t)$, $b(t)$, $A(t)$,

*At least the ones that we have considered above. There are also, of course, conditions related to the transversality conditions (6) and (7). These are examined in Section 4 below.

and $B(t)$ are all constant) in (1). This is precisely the case of (1) previously studied by others [4], [7], [16]. In the case of constant coefficients, there is no question about the optimality of a policy which results in the annihilation of all enemy aircraft during the appropriate phase of the air war (see Section 5 below). However, in the case of time-dependent returns from ground support and time-dependent Lanchester attrition-rate coefficients, the optimality of such a policy is not always guaranteed. Furthermore, the optimality of a strategy which results in the annihilation of enemy aircraft depends upon the enemy's strategy for using his aircraft if he did have them (see, for example, (20) above). More precisely, for example, the multiplier condition $\mu(t) \geq 0$ (upon which Y bases his strategy) which must hold on a constrained subarc on which $x(t) = 0$ depends upon X's optimal strategy (i.e. the enemy's strategy). This behavior is unique to differential games and has apparently never been noticed before.

Let us now give two simple examples which show the complexity in the general case of determining whether X should try to shoot down all enemy planes (during the appropriate phase of the air war). Thus, we consider the necessary condition of optimality (29) which must hold on a constrained subarc on which $y(t) = 0$ for $t_e^Y \leq t \leq t_\ell^Y$ with $t_e^Y < t_\ell^Y$. In the <u>first</u> case, let us assume that the following hold:

$$a(t) = \bar{a} = \text{constant},$$

$$b(t) = \bar{b} = \text{constant}.$$

Then, (30) takes the form

$$\text{maximum } (A(t), \bar{a}p(t)) \geq -\frac{1}{\bar{b}}\frac{dB}{dt} .$$

Let us assume that we have $A(t) \geq \bar{a}p(t)$. Then we have for the appropriate phase of the air war

(a) if $\frac{dB}{dt} \geq 0$, X should try to shoot down all the Y planes,

(b) if $\frac{dB}{dt} < 0$, X may want to fly all ground support missions now and shoot down enemy planes later in this case in which X's rate of return from ground support is decreasing.

In the <u>second</u> case, let us assume that the following hold:

$$A(t) = \bar{A} = \text{constant},$$

$$B(t) = \bar{B} = \text{constant}.$$

Then, (30) takes the form that we must have

$$\text{maximum } (\bar{A}, a(t)p(t)) \geq \frac{\bar{B}}{\{b(t)\}^2} \frac{db}{dt},$$

in order for it to be optimal to have $y(t) = 0$ for a finite interval of time. Then we have for the appropriate phase of the air war

(a) if $\frac{dB}{dt} \leq 0$, X should try to shoot down all the Y planes,

(b) if $\frac{dB}{dt} > 0$, X may want to forget about shooting down Y aircraft today because tomorrow he will be so much more effective doing this.

3.4. <u>Introduction of the Switching Functions $S_u(t)$ and $S_v(t)$.</u>

It is convenient to consider the <u>switching functions</u> $S_u(t)$ and $S_v(t)$ defined by

$$S_u(t) = (-q(t))b(t) - B(t), \tag{33}$$

and

$$S_v(t) = p(t)a(t) - A(t). \tag{34}$$

Then, we may re-write the extremal control laws (11) and (12) for the strategic variables as

for $y > 0$:

$$u^*(t) = \begin{cases} 1 & \text{for } S_u(t) < 0, \\ 0 & \text{for } S_u(t) > 0, \end{cases} \tag{35}$$

and

for $x > 0$:

$$v^*(t) = \begin{cases} 1 & \text{for } S_v(t) < 0, \\ 0 & \text{for } S_v(t) > 0. \end{cases} \tag{36}$$

Since we will develop the solution to (1) by working backwards from the end at $t_f = T$, it is convenient to introduce the <u>backwards time</u> τ defined by

$$\tau = T - t. \tag{37}$$

It should be noted that $\frac{dp}{dt} = \frac{dp}{d\tau} \cdot \frac{d\tau}{d\tau} = -\frac{dp}{d\tau}$. We have then from (33) and (34)

$$S_u(\tau) = (-q(\tau))\hat{b}(\tau) - \hat{B}(\tau), \tag{38}$$

and

$$S_v(\tau) = p(\tau)\hat{a}(\tau) - \hat{A}(\tau), \tag{39}$$

where $\hat{A}(\tau) = A(T-\tau)$, etc. and $\hat{p}(\tau) = p(T-\tau)$ and we again denote $\hat{p}(\tau)$ as $p(\tau)$, etc. Then from (35)

for $y > 0$:

$$u^*(\tau) = \begin{cases} 1 & \text{for } S_u(\tau) < 0, \\ 0 & \text{for } S_u(\tau) > 0, \end{cases} \tag{40}$$

and it follows from (38) and the adjoint equations that

$$\frac{dS_u}{d\tau} = \begin{cases} \hat{b}(\tau)\{\hat{A}(\tau)v^* + (1-v^*)\hat{a}(\tau)p(\tau)\} + (-q(\tau)\frac{d\hat{b}}{d\tau} - \frac{d\hat{B}}{d\tau} & \text{for } y > 0, \\ 0 \quad \text{for } y = 0 \quad \text{(for a finite interval of time).} \end{cases} \tag{41}$$

Similarly,

for $x > 0$:

$$v^*(\tau) = \begin{cases} 1 & \text{for } S_v(\tau) < 0, \\ 0 & \text{for } S_v(\tau) > 0, \end{cases} \tag{42}$$

and it follows from (39) and the adjoint equations that

$$\frac{dS_v}{d\tau} = \begin{cases} \hat{a}(\tau)\{\hat{B}(\tau)u^* + (1-u^*)\hat{b}(\tau)(-q(\tau))\} + p(\tau)\frac{d\hat{a}}{d\tau} - \frac{d\hat{A}}{d\tau} & \text{for } x > 0, \\ 0 \quad \text{for } x = 0 \quad \text{(for a finite interval of time).} \end{cases}$$

4. <u>Extremal End States</u>.

By an extremal end state we mean a terminal point in the state space for which necessary (transversality) conditions of optimality are satisfied. Let us first consider in detail the situation for the X aircraft. There are two possibilities which we denote as follows:

(a) $E_1^X : x(T) > 0$,

and (b) $E_2^X : x(T) = 0$.

For E_2^X, there are two further subcases as follows:

(1) $x(T) = 0$ with $x(t) = 0$ for $t_e^X \le t \le T$ where $t_e^X < T$,

(2) $x(T) = 0$ with $x(t) > 0$ for $T - \delta \le t < T$ where $\delta > 0$.

Let us first consider the end state $\underline{E_1^X : x(T) > 0}$. By (6) at $t_f = T$, we have $p(t=t_f) = v_X$, since $x(t_f=T) > 0$. Thus,

$$\text{for } x(t_f) > 0 : \quad p(t=t_f) = v_X. \tag{44}$$

Also, by (36) (or equivalently (12))

$$v^*(t=t_f) = \begin{cases} 1 & \text{for } v_X \le \dfrac{A(T)}{a(T)}, \\ 0 & \text{for } v_X \ge \dfrac{A(T)}{a(T)}. \end{cases} \tag{45}$$

<u>Comment</u>: If $v_X = 0$, then $v^*(t=t_f) = 1$. This is the only case considered in previous analyses [4], [7], [16].

Let us now consider necessary conditions of optimality for the end state $\underline{E_2^X : x(T) = 0}$. As above, there are two subcases to be considered. In $\underline{\text{case (1): } x(T) = 0 \text{ with } x(t) = 0 \text{ for } t_e^X \le t \le T \text{ where } t_e^X < T}$ (i.e. at the end the system has been on a constrained subarc for a finite interval of time), we have by (18) that

$$p(T) = \frac{A(T)}{a(T)}, \tag{46}$$

since the system has been on a constrained subarc for a finite interval of time. Also, by the transversality condition (6), we have that

$$p(T) = v_X - \nu_1 \quad \text{where} \quad \nu_1 \ge 0.$$

By (46) and (47), we have $\nu_1 = v_X - \frac{A(T)}{a(T)} \geq 0$, so that we must have

$$v_X \geq \frac{A(T)}{a(T)}, \tag{48}$$

for it to be optimal to have $x(T) = 0$ when the system has been on a constrained subarc for a finite interval of time. In case (2): $x(T) = 0$ with $x(t) > 0$ for all $t \in [T-\delta,T)$ where $\delta > 0$, it is clear that we must have $v^*(t) = 0$† for $t \in (T-\delta_1,T) \subset (T-\delta,T)$ where $0 < \delta_1 \leq \delta$, since this is the only way that we can reach $x(T) = 0$ with $x(t) > 0$ for all $t \in [T-\delta,T)$. By (10) (see also (12)) in order to have $v^*(t) = 0$ for all $t \in (T-\delta_1,T)$ we must necessarily have $p(t) \geq \frac{A(t)}{a(t)}$. By the continuity of $p(t)$ (see Appendix A) it is necessary that we have

$$p(T) \geq \frac{A(T)}{a(T)}. \tag{49}$$

We also have the transversality condition (47) so that $p(T) = v_X - \nu_1 \geq A(T)/a(T)$ which we may also write as $v_X \geq A(T)/a(T) + \nu_1 \geq A(T)/a(T)$, since $\nu_1 \geq 0$. Thus, we see that (48) is again a necessary condition for it to be optimal to have $x(T) = 0$ but $x(t) > 0$ for all $t \in [T-\delta,T)$ where $\delta > 0$.

Considering the above, we see that we have proved the following theorem.

THEOREM 1: If an optimal strategy for Y is to result in $x^*(T) = 0$, then it is necessary that

$$v_X \geq \frac{A(T)}{a(T)}.$$

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†It should be recalled that we have shown it to be impossible to have a singular solution for $x,y > 0$. Hence, $v^*(t)$ must be equal to 0 or 1 almost everywhere in time.

In other words, in order that the end state $\underline{E_2^X : x(T) = 0}$ be an extremal end state we must have $v_X \geq A(T)/a(T)$. We also have the following as an immediate corollary.

> COROLLARY 1.1: If $v_X < \frac{A(T)}{a(T)}$, then an optimal strategy for Y leads to $x^*(T) > 0$.

<u>Remark</u>: In the problem considered by Isaacs [7] and others [4], [16] previously, we have $v_X = 0$, $A(t) = 1$, and $a(t) = \bar{a} =$ constant so that an optimal strategy for Y resulted in $x^*(T) > 0$. The optimality of this condition apparently was never examined previously. In fact, an implicit assumption in Friedman's example on pp. 239-240 of [4] is that an optimal policy results in $x^*(T) > 0$. From Theorem 1 we see that this need not be true in general.

In a similar fashion, we can prove the following theorem.

> THEOREM 2: If an optimal strategy for X is to result in $y^*(T) = 0$, then it is necessary that
> $$v_Y \geq \frac{B(T)}{b(T)}.$$

We also have the following as an immediate corollary.

> COROLLARY 2.1: If $v_Y < \frac{B(T)}{b(T)}$, then an optimal strategy for X leads to $y^*(T) > 0$.

Remark: It seems appropriate to point out the following interpretations:

(a) $a(T)v_X$ = rate at which one Y aircarft can destroy value of X aircraft at end of campaign,

(b) $A(T)$ = rate of return from one Y aircraft from flying ground support missions at end.

Considering the above, the meaning of Theorem 1 is clear: in order for Y to follow a strategy which results in $x(T) = 0$, it must be to his advantage to do so.

5. Solution for Special Case of Constant Coefficients.

In this section we will develop the solution to (1) for the special case in which all the coefficients are constants. In other words, we will consider the case in which the following hold:

$$a(t) = \bar{a} = \text{constant},$$

$$b(t) = \bar{b} = \text{constant},$$

$$A(t) = \bar{A} = \text{constant},$$

$$B(t) = \bar{B} = \text{constant}.$$

For notational convenience we will again denote $\bar{a}$ as a, etc. It is then convenient to re-state the problem as follows:

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\ \{v_X x(t_f) - v_Y y(t_f) + \int_0^{t_f} (Bux - Avy)dt\},$$

with stopping rule: $t_f - T = 0,$

subject to: (air-battle dynamics)
$$\frac{dx}{dt} = r - (1-v)ay, \tag{50}$$

$$\frac{dy}{dt} = s - (1-u)bx,$$

with initial conditions $x(t=0) = x_0$ and $y(t=0) = y_0$,

and

$$x,y \geq 0 \quad \text{(State Variable Inequality Constraints)},$$

$$0 \leq u,v \leq 1 \quad \text{(Strategic Variable Inequality Constraints)},$$

where a is a given constant and similarly for b, A, and B.

Let us now review the necessary conditions of optimality that we have developed for (50) in Section 3. Let us recall the <u>switching functions</u> in <u>backwards time</u> $\tau = T - t$. These are

$$S_u(\tau) = b(-q(\tau)) - B, \tag{51}$$

and

$$S_v(\tau) = ap(\tau) - A. \tag{52}$$

We had developed earlier (see (40) through (43)) the following characterization of optimal strategies:

for $y > 0$:

$$u^*(\tau) = \begin{cases} 1 & \text{for } S_u(\tau) < 0, \\ 0 & \text{for } S_u(\tau) > 0, \end{cases} \tag{53}$$

and

for $x > 0$:

$$v^*(\tau) = \begin{cases} 1 & \text{for } S_v(\tau) < 0, \\ 0 & \text{for } S_v(\tau) > 0, \end{cases} \tag{54}$$

with

$$\frac{dS_u}{d\tau} = \begin{cases} b\{A_v^* + (1-v^*)ap(\tau)\} > 0 & \text{for } y > 0, \\ 0 \quad \text{for } y = 0 & \text{(for a finite interval of time)}, \end{cases} \tag{55}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} a\{Bu^* + (1-u^*)b(-q(\tau))\} > 0 & \text{for} \quad x > 0, \\ \\ 0 \quad \text{for} \quad x = 0 \quad \text{(for a finite interval of time).} \end{cases} \tag{56}$$

Thus we see that $\frac{dS_u}{d\tau} \geq 0$ with strict inequality for $y > 0$ and similarly for $S_v(\tau)$. From (15) we see that singular subarcs are impossible.

On a constrained subarc on which $x(t) = 0$ for $t_e^X \leq t \leq t_\ell^X$ we have shown the following to be necessary conditions for optimality:

$$v^*(t) = 1 - \frac{r}{ay(t)} \quad \text{for} \quad t_e^X < t < t_\ell^X \tag{57}$$

$$r < ay(t) \quad \text{for} \quad t_e^X \leq t \leq t_\ell^X, \tag{58}$$

$$p(t) = \frac{A}{a} \quad \text{for} \quad t_e^X \leq t \leq t_\ell^X. \tag{59}$$

The requirement (21) is automatically satisfied, since

$$\text{maximum } \{B, b(-q(t))\} \geq B > 0 = -\frac{d}{dt}\left\{\frac{A}{a}\right\} \quad \text{for} \quad t_e^X < t < t_\ell^X.$$

Similar consitions hold on a constrained subarc on which $y(t) = 0$ for a finite interval of time.

Concerning extremal end states, we have developed the following.

PROPOSITION 1: If an optimal strategy for Y is to result in $x^*(T) = 0$, then it is necessary that

$$v_X \geq \frac{A}{a}.$$

PROPOSITION 2: If $v_X < \frac{A}{a}$, then an optimal strategy for Y leads to $x^*(T) > 0$.

PROPOSITION 3: If an optimal strategy for X is to result in $y^*(T) = 0$, then it is necessary that

$$v_Y \geq \frac{B}{b}.$$

PROPOSITION 4: If $v_Y < \frac{B}{b}$, then an optimal strategy X leads to $y^*(T) > 0$.

We also have the following boundary conditions for the adjoint variables:

$$p(t=T) = v_X - \nu_1 \quad \text{where} \quad \nu_1 \begin{cases} = 0 & \text{for } x(T) > 0, \\ \geq 0 & \text{for } x(T) = 0, \end{cases} \tag{60}$$

$$q(t=T) = v_Y + \nu_2 \quad \text{where} \quad \nu_2 \begin{cases} = 0 & \text{for } y(T) > 0, \\ \geq 0 & \text{for } y(T) = 0. \end{cases} \tag{61}$$

We may then write the differential equations for the switching functions as

$$\frac{dS_u}{d\tau} = \begin{cases} b\{A + (1-v^*)S_v(\tau)\} & \text{for } y > 0, \\ 0 & \text{for } y = 0 \text{ (for a finite interval of time),} \end{cases} \tag{62}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} a\{B + (1-u^*)S_u(\tau)\} & \text{for } x > 0, \\ 0 & \text{for } x = 0 \text{ (for a finite interval of time),} \end{cases} \tag{63}$$

with initial conditions

$$S_u(\tau=0) = (bv_Y - B) - b\nu_2 \quad \text{and} \quad S_v(\tau=0) = (av_X - A) - a\nu_1. \tag{64}$$

5.1. Synthesis of Extremal Strategic-Variable Pair.

There are four cases to be considered:

(I) $v_X < \frac{A}{a}$ (II) $v_X \geq \frac{A}{a}$

(1) $v_Y < \frac{B}{b}$ (1) $v_Y < \frac{B}{b}$

(2) $v_Y \geq \frac{B}{b}$ (2) $v_Y \geq \frac{B}{b}$.

These cases will be denoted as (I1), (I2), (II1), and (II2). The solution to (50) must be developed for each case separately. It should be noted that Case (I2) and Case (II1) are symmetric: Case (I2) is the same as Case (II1) only with the roles of X and Y interchanged. These then are three cases of different structures for the optimal strategies.

In each case, the solution to (50) is developed by working backwards from the end of the war at $t = t_f = T$ (or, equivalently, $\tau = 0$) where the boundary information (60) through (63) is known. The differential equations for the switching functions $S_u(\tau)$ and $S_v(\tau)$ may then be integrated backwards (using (53) and (54) (or (57) on constrained subarcs)) so that the initial conditions are just met. This procedure leads to synthesis of an extremal strategic-variable pair and a corresponding extremal trajectory.* In the case in which the extremal trajectory is unique (in the sense that there is a single extremal path through any point of the state space which leads to the terminal manifold, it is the optimal trajectory. This is the situation in all cases except for one exceptional set of circumstances for Case (I2). The optimal strategies in this (exceptional) situation have not been resolved at the time of the writing of this report. This is an important future research task, since the development of

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*By an extremal trajectory we mean a path along which the necessary conditions of optimality (see Appendix A) are satisfied everywhere in time.

computational schemes for time-sequential combat games (such as application of Lagrange dynamic programming as proposed in [13], [14]) depends upon its successful completion.

Case (II): $v_X < \frac{A}{a}$ and $v_Y < \frac{B}{b}$.

Let us make the nonrestrictive assumption that

$$\frac{A-av_X}{aB} < \frac{B-bv_Y}{bA} . \tag{65}$$

By Propositions 2 and 4 we have that $x^*(T) > 0$ and $y^*(T) > 0$ so that by (64) we have

$$S_u(\tau=0) = bv_Y - B < 0 \quad \text{and} \quad S_v(\tau=0) = av_X - A < 0. \tag{66}$$

One obtains from (66) via (53) and (54) that

$$u^*(\tau=0) = 1 \quad \text{and} \quad v^*(\tau=0) = 1. \tag{67}$$

By straightforward continuity arguments, it is readily seen that

$$\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases} \quad \text{for} \quad \tau \in [0,\tau_1], \tag{68}$$

where τ_1 denotes the "backwards time" of the first switch in (extremal) tactics. Using (62), (63), and (68), we see that the differential equations for the switching functions are for $0 \le \tau \le \tau_1$

$$\frac{dS_u}{d\tau} = bA \quad \text{with} \quad S_u(\tau=0) = bv_Y - B < 0, \tag{69}$$

and

$$\frac{dS_v}{d\tau} = aB \quad \text{with} \quad S_v(\tau=0) = av_X - A < 0, \tag{70}$$

so that for $0 \leq \tau \leq \tau_1$

$$S_u(\tau) = bA\tau + bv_Y - B, \tag{71}$$

and

$$S_v(\tau) = aB\tau + av_X - A. \tag{72}$$

The "backwards" switching time τ_u is determined by $S_u(\tau=\tau_u) = 0$ and hence

$$\tau_u = \frac{B-bv_Y}{bA} . \tag{73}$$

Similarly, we have

$$\tau_v = \frac{A-av_X}{aB} . \tag{74}$$

The assumption (65) then yields that $\tau_1 = \tau_v < \tau_u$ so that $v^*(\tau)$ switches first in backwards time at

$$\tau_1 = \tau_v = \frac{A-av_X}{aB} . \tag{75}$$

We then have for $x(\tau) > 0$

$$\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 0 \end{cases} \quad \text{for} \quad \tau \in (\tau_1, \tau_2), \tag{76}$$

where τ_2 denotes the "backwards time" of the second switch in (extremal) tactics. It should be noted that by (50) and (76) we have $y(\tau) > 0$ for $0 \leq \tau < \tau_2$. Using (62), (63), (71), (72), (75), and (76), we see that the differential equations for the switching functions are for $\tau_1 \leq \tau \leq \tau_2$

$$\frac{dS_u}{d\tau} = b\{A+S_v(\tau)\} \quad \text{with} \quad S_u(\tau=\tau_1) = bA\left\{\left(\frac{A-av_X}{aB}\right) - \left(\frac{B-bv_Y}{bA}\right)\right\} < 0, \tag{77}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} aB & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0 \quad \text{(for finite interval of time)} \end{cases} \qquad \text{with} \quad S_v(\tau=\tau_1) = 0. \tag{78}$$

There are two cases to be considered:

(a) $x(t=T-\tau_1) > 0$,

(b) $x(t=T-\tau_1) = 0$.

In Case (a): $x(t=T-\tau_1) > 0$, we have that $x(\tau) > 0$ for $0 \le \tau \le T$ by (50) and (76). Hence, (54) and (78) yield that $v^*(\tau) = 0$ for all $\tau > \tau_1$. Integrating (77) and (78), we obtain

$$S_u(\tau) = \frac{ab}{2B}(B\tau+v_X)^2 - \frac{ab}{2B}\left(\frac{A}{a}\right)^2 + bA\left\{\left(\frac{A-av_X}{aB}\right)-\left(\frac{B-bv_Y}{bA}\right)\right\}, \tag{79}$$

and

$$S_v(\tau) = aB(\tau-\tau_1) = aB\tau - (A-av_X). \tag{80}$$

The switching time $\tau_u = \tau_2$ is determined by $S_u(\tau=\tau_u) = 0$ and hence

$$\tau_2 = \tau_u = \frac{1}{aB}\left\{\sqrt{A^2 + 2aAB\left\{\left(\frac{B-bv_Y}{bA}\right)-\left(\frac{A-av_X}{aB}\right)\right\}} - av_X\right\} > \tau_1. \tag{81}$$

Again, (53) and (55) yield that once $u^*(\tau) = 0$ there are no further switches in the strategic variables as we progress backwards in time.

In Case (b): $x(t=T-\tau_1) = 0$, we have that $x(\tau) = 0$ for $\tau_\ell^X = \tau_1 \le \tau \le \tau_e^X$ and $x(\tau) > 0$ for $\tau_e^X < \tau \le \tilde{\tau}_2$, where $\tilde{\tau}_2$ denotes the "backwards time" of the second switch in (extremal) tactics in the case in which $x(\tau) = 0$ for $\tau_1 \le \tau \le \tau_e^X \le \tilde{\tau}_2$. When $x(\tau) = 0$ for $\tau_\ell^X = \tau_1 \le \tau \le \tau_e^X \le \tilde{\tau}_2$, we obtain from (77) and (78) that

$$S_u(\tau) = bA\left\{\tau - \left(\frac{B-bv_Y}{bA}\right)\right\}, \tag{82}$$

and

$$S_v(\tau) = 0. \tag{83}$$

When $x(\tau) > 0$ for $\tau_e^X < \tau \leq \tau_2$, (77) and (78) become for $\tau_e^X \leq \tau \leq \tilde{\tau}_2$

$$\frac{dS_u}{d\tau} = b\{A + S_v(\tau)\} \quad \text{with} \quad S_u(\tau=\tau_e^X) = bA\left\{\tau_e^X - \left(\frac{B-bv_Y}{bA}\right)\right\}, \tag{84}$$

and

$$\frac{dS_v}{d\tau} = aB \quad \text{with} \quad S_v(\tau=\tau_e^X) = 0. \tag{85}$$

Integration of (84) and (85) yields that for $\tau_e^X \leq \tau \leq \tilde{\tau}_2$

$$S_u(\tau) = \frac{abB}{2}(\tau-\tau_e^X)^2 + bA(\tau-\tau_e^X) + bA\left\{\tau_e^X - \left(\frac{B-bv_Y}{bA}\right)\right\}, \tag{86}$$

and

$$S_v(\tau) = aB(\tau-\tau_e^X). \tag{87}$$

The switching time $\tau_u = \tilde{\tau}_2$ is determined by $S_u(\tau=\tau_u) = 0$ and hence

$$\tilde{\tau}_2 = \tau_u = \tau_e^X + \frac{1}{aB}\left\{\sqrt{A^2-2aAB\left\{\tau_e^X-\left(\frac{B-bv_Y}{bA}\right)\right\}} - A\right\}, \tag{88}$$

for $\left(\frac{A-av_X}{aB}\right) = \tau_1 \leq \tau_e^X \leq \tilde{\tau}_2' = \left(\frac{B-bv_Y}{bA}\right)$. Thus, we have $\tilde{\tau}_2 = \tilde{\tau}_2(\tau_e^X)$. It is readily seen that $\tilde{\tau}_2$ is a strictly increasing function of τ_e^X for $\tau_1 \leq \tau_e^X < \left(\frac{B-bv_Y}{bA}\right)$, since we have

$$\frac{\partial\tilde{\tau}_2}{\partial\tau_e^X} = 1 - \frac{A}{\sqrt{A^2 - 2aAB\left\{\tau_e^X - \left(\frac{B-bv_Y}{bA}\right)\right\}}}. \tag{89}$$

It is readily seen that $\frac{\partial \tilde{\tau}_2}{\partial \tau_e^X} > 0$ for $\tau_e^X < \left(\frac{B-bv_Y}{bA}\right)$, that $\frac{\partial \tilde{\tau}_2}{\partial \tau_e^X} = 0$ for $\tau_e^X = \left(\frac{B-bv_Y}{bA}\right)$, and that $\frac{\partial \tilde{\tau}_2}{\partial \tau_e^X} < 0$ for $\tau_e^X > \left(\frac{B-bv_Y}{bA}\right)$. Hence, $\tilde{\tau}_2$ has a maximum (which we denote as $\tilde{\tau}_2'$) for $\tau_e^X = \left(\frac{B-bv_Y}{bA}\right)$. It seems appropriate to note that

$$\tilde{\tau}_2 = \tau_2 = \frac{1}{aB}\left\{\sqrt{A^2 + 2aAB\left\{\left(\frac{B-bv_Y}{bA}\right)-\left(\frac{A-av_X}{aB}\right)\right\}} - av_X\right\} \quad \text{for} \quad \tau_e^X = \tau_1, \qquad (90)$$

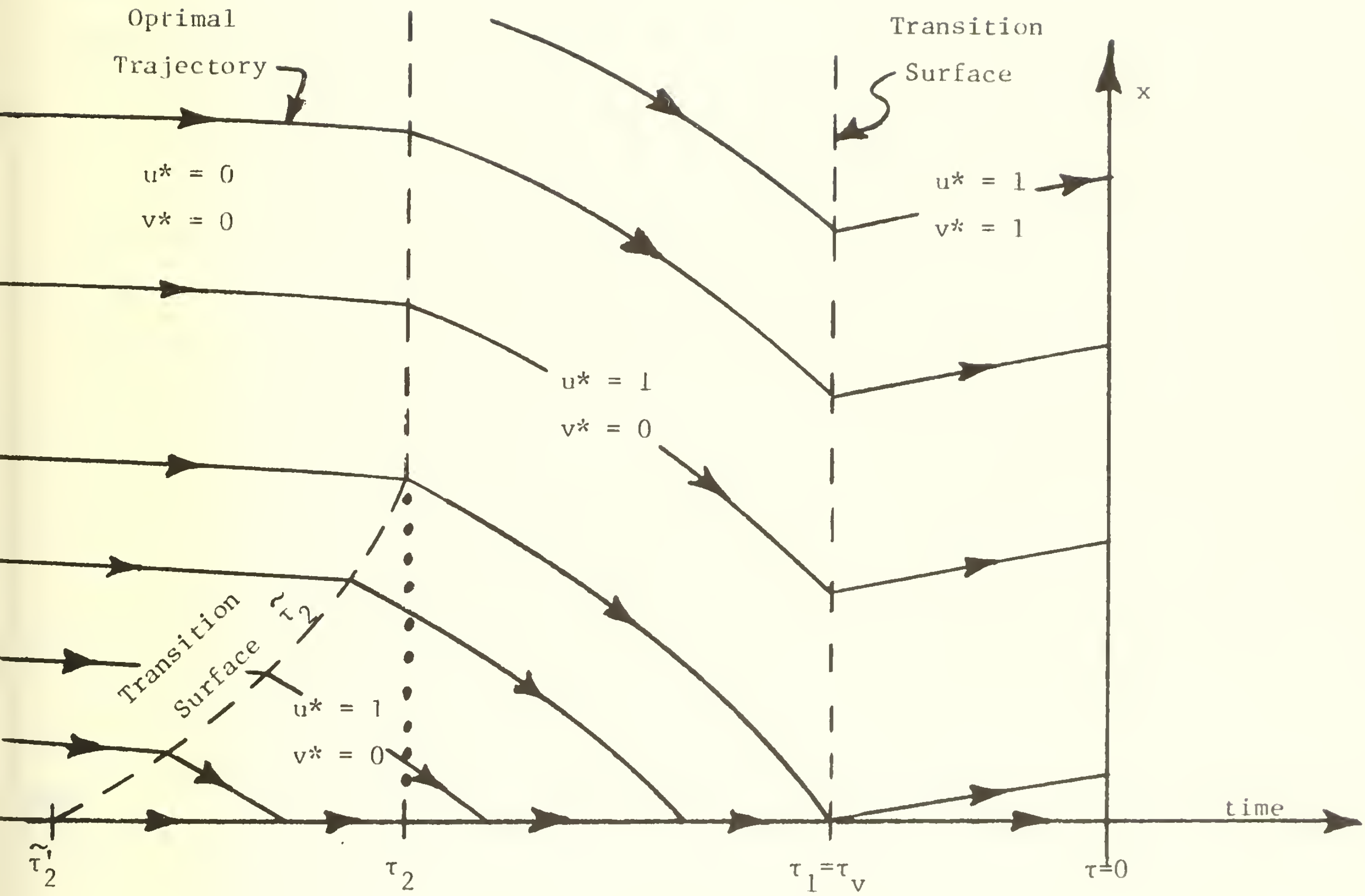
and

$$\tilde{\tau}_2 = \tilde{\tau}_2' = \left(\frac{B-bv_Y}{bA}\right) \quad \text{for} \quad \tau_e^X = \left(\frac{B-bv_Y}{bA}\right).$$

Thus, we see that

$$\tau_2 < \tilde{\tau}_2 < \tilde{\tau}_2' = \left(\frac{B-bv_Y}{bA}\right), \qquad (92)$$

for $\left(\frac{A-av_X}{aB}\right) = \tau_1 < \tau_e^X < \left(\frac{B-bv_Y}{bA}\right)$. This means that the switching time for $u^*(\tau)$ depends upon force levels, since it depends upon whether or not $x(t)$ is driven to zero before $t = T - \tau_1 = T - \left(\frac{A-av_X}{aB}\right)$ when $T > \tau_1$. Furthermore, when $x(t) = 0$ for $T - \tau_e^X = t_e^X \leq t \leq t_\ell^X = T - \tau_1$, the switching time for X, i.e. $\tilde{\tau}_2 = \tau_u$, depends upon the time (or, equivalently, the backwards time τ_e^X) at which the X forces are annihilated. Again, (53) and (55) yield that once $u^*(\tau) = 0$ there are no further switches in the strategic variables as we progress further backwards in time. This situation is shown diagrammatically in Figure 3 below. Thus, we see that X switches to concentration on ground support earlier in those compaigns in which x_0, y_0, and T are such that X will lose all his planes before



Note:	<u>Strategic Variables</u>
	X: u
	Y: v

Figure 3. Dependence of X Switching Time $\tilde{\tau}_2 = \tau_u$ Upon Force Levels (<u>not drawn to scale</u>).

$T - \left(\frac{A-av_X}{aB}\right) > 0$ but after $T - \left(\frac{B-bv_Y}{bA}\right) > 0.$* This means that X's optimal strategy is to use his planes for ground support earlier in such cases than when $x(t=T-\tau_1) > 0$ because they are annihilated before the last phase of the air war (when both sides concentrate on ground support). Thus, X uses his planes for ground support while they are still available. This phenomenon was first noted (for a simpler case of (50)) by Professor Rufus Isaacs [7] (see also pp. 948-949 of [9]).**

Case (I2): $v_X < \frac{A}{a}$ and $v_Y \geq \frac{B}{b}$.

By Proposition 2 we have that $x^*(T) > 0$ so that by (64) we have

$$S_v(\tau=0) = av_X - A < 0, \tag{93}$$

and by (54) we obtain

$$v^*(\tau=0) = 1. \tag{94}$$

By Proposition 3 an optimal strategy for X can lead to $y^*(T) = 0$ so that by (64) we have

$$S_u(\tau=0) = (bv_Y-B) - b\nu_2 \quad \text{where} \quad \nu_2 \begin{cases} = 0 & \text{for } y(T) > 0, \\ \geq 0 & \text{for } y(T) = 0. \end{cases} \tag{95}$$

There are two cases to be considered:

(a) $y(t=T) > 0$,

(b) $y(t=T) = 0$.

In Case (a): $y(t=T) > 0$, we have that

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*The reader should recall the nonrestrictive assumption (65).

**This behavior is not noted in either [4] (see pp. 239-240) or [16].

$$S_u(\tau=0) = bv_Y - B \geq 0. \tag{96}$$

By continuity of $y(t)$, we have that $y(t) > 0$ for $t \in (T-\delta, T]$ where $\delta > 0$ so that (55) and (96) yield that $S_u(\tau) > 0$ for $0 < \tau \leq T$. Hence, (53) yields that $u^*(\tau) = 0$ for $0 \leq \tau \leq T$, and a backwards integration of the state equation for $y(\tau)$ yields $y(\tau) > 0$ for $0 \leq \tau \leq T$. Also, it is readily seen by a straightforward continuity argument on $S_v(\tau)$ that

$$\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 1 \end{cases} \quad \text{for} \quad \tau \in [0, \tau_1], \tag{97}$$

where τ_1 denotes the "backwards time" of the first switch in (extremal) tactics. It is readily seen that $x(\tau) > 0$ for $\tau \in [0, \tau_1)$. The differential equation for the switching functions are then for $0 \leq \tau \leq \tau_1$

$$\frac{dS_u}{d\tau} = bA \qquad \text{with} \quad S_u(\tau=0) = bv_Y - B \geq 0, \tag{98}$$

$$\frac{dS_v}{d\tau} = a\{B + S_u(\tau)\} \quad \text{with} \quad S_v(\tau=0) = av_X - A < 0, \tag{99}$$

so that for $0 \leq \tau \leq \tau_1$

$$S_u(\tau) = bA\tau + bv_Y - B, \tag{100}$$

and

$$S_v(\tau) = \frac{ab}{2A}(A\tau + v_Y)^2 - \frac{ab}{2A}v_Y^2 + av_X - A. \tag{101}$$

The switching time $\tau_v = \tau_1$ is determined by $S_v(\tau = \tau_v) = 0$ and hence

$$\tau_1 = \tau_v = \frac{1}{A}\left\{\sqrt{v_Y^2 + \frac{2A}{ab}(A - av_X)} - v_Y\right\}. \tag{102}$$

There are two further subcases to be considered:

(1) $\quad x(t=T-\tau_1) > 0,$

(2) $\quad x(t=T-\tau_1) = 0.$

In Subcase (1): $x(t=T-\tau_1) > 0$, continuity arguments on $x(t)$ lead to $S_v(\tau) > 0$ for $\tau > \tau_1$. Hence

$$\left\{\begin{array}{l} u^*(\tau) = 0 \\ \\ v^*(\tau) = 0 \end{array}\right. \qquad \text{for} \qquad \tau \in (\tau_1, T], \tag{103}$$

and $x(t) > 0$ for $0 \quad t \quad T$. In Subcase (2): $x(t=T-\tau_1) = 0$, we have that $x(\tau) = 0$ for $\tau_\ell^X = \tau_1 \le \tau \le \tau_e^X \le T$ and $x(\tau) > 0$ for $\tau_e^X < \tau \le T$. Using (62) and (63), we readily obtain for $\tau_1 \le \tau \le \tau_e^X$

$$S_u(\tau) = bA\tau + bv_Y - B > 0, \tag{104}$$

and

$$S_v(\tau) = 0. \tag{105}$$

Thus we have

$$\left\{\begin{array}{l} u^*(\tau) = 0 \\ \\ v^*(\tau) = 1 - \dfrac{r}{ay} \end{array}\right. \qquad \text{for} \qquad \tau \in (\tau_1, \tau_e^X]. \tag{106}$$

Since $x(\tau) > 0$ for $\tau \in (\tau_e^X, \tau_e^X+\delta)$, arguments similar to the above yield that $S_v(\tau) > 0$ and $x(\tau) > 0$ for $\tau_e^X < \tau \le T$. Hence

$$\left\{\begin{array}{l} u^*(\tau) = 0 \\ \\ v^*(\tau) = 0 \end{array}\right. \qquad \text{for} \qquad \tau \in (\tau_e^X, T]. \tag{107}$$

In Case (b): $y(t=T) = 0$, there are two further subcases to be considered:

(1) $y(t) = 0$ for $t_e^Y \le t \le T$ with $t_e^Y < T$,

(2) $y(t{=}T) = 0$ but $y(t) > 0$ for $T - \delta < t < T$ where $\delta > 0$.

In Subcase (1): $y(t) = 0$ for $t_e^Y \le t \le T$ with $t_e^Y < T$, it is readily seen that we must have

$$\begin{cases} u^*(\tau) = 1 - \dfrac{s}{bx} \\ \\ v^*(\tau) = 1 \end{cases} \qquad \text{for} \qquad \tau \in [0, \tau_e^Y], \tag{108}$$

so that a result analogous to (59), (62), and (63) yield that for $0 \le \tau \le \tau_e^Y \le \tilde{\tau}_1$

$$S_u(\tau) = 0, \tag{109}$$

and

$$S_v(\tau) = aB\tau + av_X - A. \tag{110}$$

It is readily seen that we must have $y(\tau) > 0$ for $\tau_e^Y < \tau \le T$. Thus

$$\begin{cases} u^*(\tau) = 0 \\ \\ v^*(\tau) = 1 \end{cases} \qquad \text{for} \quad \tau \in (\tau_e^Y, \tilde{\tau}_1], \tag{111}$$

where $\tilde{\tau}_1$ denotes the "backwards time" of the first switch in (extremal) tactics in the case in which $y(\tau) = 0$ for $0 \le \tau \le \tau_e^Y$, and

$$\begin{cases} u^*(\tau) = 0 \\ \\ v^*(\tau) = 0 \end{cases} \qquad \text{for} \quad \tau \in (\tilde{\tau}_1, T], \tag{112}$$

by (54) and (56), since $y(\tau) > 0$ for $\tau > \tilde{\tau}_1 \ge \tau_e^Y$. Then (62) and (63) become for $\tau_e^Y \le \tau \le \tilde{\tau}_1$

$$\frac{dS_u}{d\tau} = bA \qquad \text{with} \quad S_u(\tau=\tau_e^Y) = 0, \tag{113}$$

$$\frac{dS_v}{d\tau} = a\{B+S_u(\tau)\} \quad \text{with} \quad S_v(\tau=\tau_e^Y) = aB\tau_e^Y + av_X - A \le 0, \tag{114}$$

so that for $\tau_e^Y \le \tau \le \tilde{\tau}_1$

$$S_u(\tau) = bA(\tau-\tau_e^Y), \tag{115}$$

and

$$S_v(\tau) = \frac{abA}{2}(\tau-\tau_e^Y)^2 + aB(\tau-\tau_e^Y) + aB\tau_e^Y + av_X - A. \tag{116}$$

The switching time $\tau_v = \tilde{\tau}_1$ is determined by $S_v(\tau=\tau_v) = 0$ and hence

$$\tilde{\tau}_1 = \tau_v = \tau_e^Y + \frac{1}{bA}\left\{\sqrt{B^2 - 2bAB\left\{\tau_e^Y - \left(\frac{A-av_X}{aB}\right)\right\}} - B\right\}, \tag{117}$$

for $0 < \tau_e^Y \le \tilde{\tau}_1' = \left(\frac{A-av_X}{aB}\right)$. Thus, we have $\tilde{\tau}_1 = \tilde{\tau}_1(\tau_e^Y)$. It is readily seen that $\tilde{\tau}_1$ is a strictly increasing function of τ_e^Y for $0 < \tau_e^Y < \left(\frac{A-av_X}{aB}\right)$, since we have

$$\frac{\partial\tilde{\tau}_1}{\partial\tau_e^Y} = 1 - \frac{B}{\sqrt{B^2 - 2bAB\left\{\tau_e^Y - \left(\frac{A-av_X}{aB}\right)\right\}}} . \tag{118}$$

It is readily seen that $\frac{\partial\tilde{\tau}_1}{\partial\tau_e^Y} > 0$ for $\tau_E^Y < \left(\frac{A-av_X}{aB}\right)$, that $\frac{\partial\tilde{\tau}_1}{\partial\tau_e^Y} = 0$ for $\tau_e^Y = \left(\frac{A-av_X}{aB}\right)$, and that $\frac{\partial\tilde{\tau}_1}{\partial\tau_e^Y} < 0$ for $\tau_e^Y > \left(\frac{A-av_X}{aB}\right)$. Hence, $\tilde{\tau}_1$ has a maximum (which we denote as $\tilde{\tau}_1'$) for $\tau_e^Y = \left(\frac{A-av_X}{aB}\right)$. It seems appropriate to note that

$$\tilde{\tau}_1^0 = \lim_{\tau_e^Y\to 0} \tilde{\tau}_1 = \frac{1}{bA}\left\{\sqrt{B^2 + 2bAB\left(\frac{A-av_X}{aB}\right)} - B\right\}, \tag{119}$$

and

$$\tilde{\tau}_1 = \tilde{\tau}_1' = \left(\frac{A-av_X}{aB}\right) \qquad \text{for} \quad \tau_e^Y = \left(\frac{A-av_X}{aB}\right) . \tag{120}$$

It should be noted that

$$\tilde{\tau}_1^0 < \tilde{\tau}_1 < \tilde{\tau}_1' - \left(\frac{A-av_X}{aB}\right) , \tag{121}$$

for $0 < \tau_e^Y < \left(\frac{A-av_X}{aB}\right)$. As we discussed for Case (I1): $v_X < \frac{A}{a}$ and $v_Y < \frac{B}{b}$, this means that the switching time for $v^*(\tau)$ depends upon force levels, since it depends upon whether or not $y(t)$ is driven to zero before $t = T$. Other aspects so far are similar to this previous case.

In <u>Subcase (2): $y(t=T) = 0$ but $y(t) > 0$ for $T - \delta < t < T$ where $\delta > 0$</u>, it is clear that we must have $u^*(\tau) = 0$ for all $\tau \in (0,\delta_1) \subset (0,\delta)$. Thus, we have

$$\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 1 \end{cases} \quad \text{for} \quad \tau \in [0,\bar{\tau}_1], \tag{122}$$

where $\bar{\tau}_1$ denotes the "backwards time" of the first switch in (extremal) tactics in the case in which $y(\tau=0) = 0$ but $y(\tau) > 0$ for $0 < \tau < \delta$. Considering (122), (64) yields that

$$S_u(\tau=0) = (bv_Y-B) - b\nu_2 \geq 0, \tag{123}$$

where

$$0 \leq \nu_2 \leq \left(\frac{bv_Y-B}{b}\right) . \tag{124}$$

Using (122) in the integration of the differential equations (62) and (63) for the switching functions* with the boundary conditions (93) and (123), we obtain for $0 \leq \tau \leq \bar{\tau}_1$

$$S_u(\tau) = bA\tau + bv_Y - B, \tag{125}$$

and

$$S_v(\tau) = \frac{ab}{2A}\{A\tau+(v_Y-\nu_2)\}^2 - \frac{ab}{2A}(v_Y-\nu_2)^2 + av_X - A. \tag{126}$$

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*It should be noted that $x(\tau), y(\tau) > 0$ for $\tau > 0$.

The switching time $\tau_v = \bar{\tau}_1$ is determined by $S_v(\tau=\tau_v) = 0$ and hence

$$\bar{\tau}_1 = \tau_v = \frac{1}{A}\left\{\sqrt{(v_Y-\nu_2)^2 + \frac{2A}{ab}(A-av_X)} - (v_Y-\nu_2)\right\}, \tag{127}$$

for $0 \le \nu_2 \le \left(\frac{bv_Y-B}{b}\right)$. It is readily seen that

$$\begin{cases} u^*(\tau) = 0 \\ v^*(\tau) = 0 \end{cases} \quad \text{for} \quad \tau \in (\bar{\tau}_1, T]. \tag{128}$$

Thus, we have $\bar{\tau}_1 = \bar{\tau}_1(\nu_2)$ where ν_2 is chosen so that $y(t=T) = 0$ but $y(t) > 0$ for $T - \delta < t < T$. It is readily seen that $\bar{\tau}_1$ is a strictly increasing function of ν_2 for $0 \le \nu_2 \le \left(\frac{bv_Y-B}{b}\right)$, since we have

$$\frac{\partial\bar{\tau}_1}{\partial\nu_2} = \frac{1}{A}\left\{1 - \frac{(v_Y-\nu_2)}{\sqrt{(v_Y-\nu_2)^2 + \frac{2A}{ab}(A-av_X)}}\right\} > 0. \tag{129}$$

It seems appropriate to note that

$$\bar{\tau}_1 = \tau_1 = \frac{1}{A}\left\{\sqrt{v_Y^2 + \frac{2A}{ab}(A-av_X)} - v_Y \quad \text{for} \quad \nu_2 = 0, \tag{130}$$

and

$$\bar{\tau}_1 = \tilde{\tau}_1^{\,o} = \frac{1}{bA}\left\{\sqrt{B+2bAB\left(\frac{A-av_X}{aB}\right)} - B\right\} \quad \text{for} \quad \nu_2 = \left(\frac{bv_Y-B}{b}\right). \tag{131}$$

Thus, $\bar{\tau}_1$ is a strictly increasing function of ν_2, and we have

$$\tau_1 \le \bar{\tau}_1 \le \tilde{\tau}_1^{\,o} \quad \text{for} \quad 0 \le \nu_2 \le \left(\frac{bv_Y-B}{b}\right). \tag{132}$$

It should also be noted that $\tilde{\tau}_1^{\,o} > \tau_1$ for $bv_Y > B$. Thus, from (132) we see that for $bv_Y > B$ more than one extremal can lead to a point $P^f = (x_f, y_f) = (x_f, 0)$ with $x_f > 0$ on the terminal manifold. Thus, in Figure 4 the extremal trajectories labelled as (1), (2), and (3) all lead to the terminal manifold with $y_f = 0$. Let $P^o = (x_0, y_0)$ denote a point in the

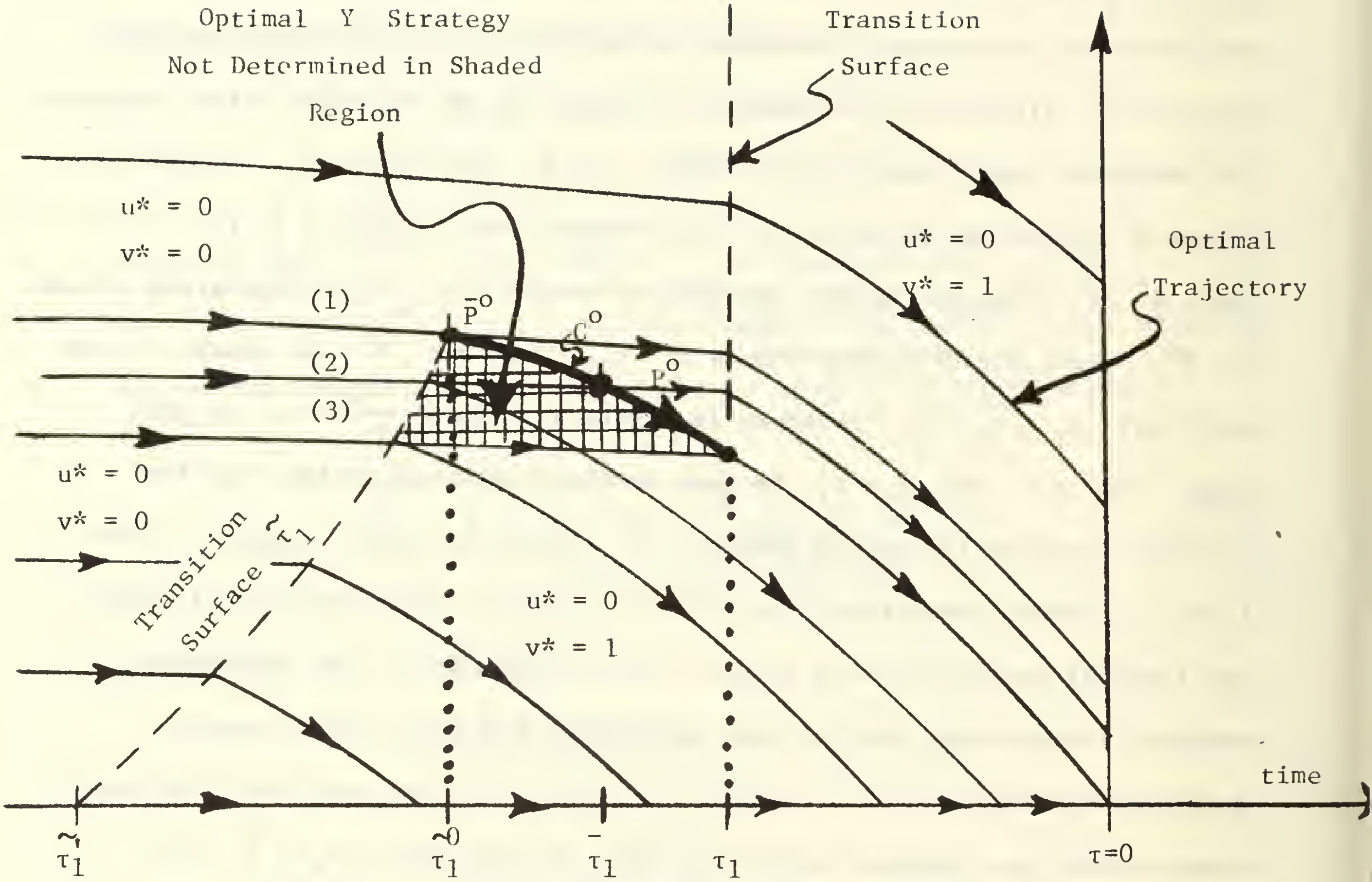
initial state space from which an extremal leads to $P^f = (x_f,0)$ with $\begin{cases} u^*(t) = 0 \\ v^*(t) = 1 \end{cases}$ for $0 \le t \le T$ and $\tau_1 \le T \le \tilde{\tau}_1^o$. Furthermore, let C^o denote the set of all such points, i.e. $C^o = \{P^o |$ system reaches $P^f = (x_f,0)$ from $\begin{cases} u^*(t) = 0 \\ v^*(t) = 1 \end{cases}$ for $0 \le t \le T$ and $\tau_1 \le T \le \tilde{\tau}_1^o\}$. From the backwards synthesis procedure for the extremal strategic-variable pair, it is readily seen that the set C^o lies on the "backwards extension" of the extremal which originates (in backwards time) from $(x_f,0)$ with $x_f > 0$. The curve C^o is shown in Figure 4*. From any point on C^o an extremal with $\begin{cases} u^*(t) = 0 \\ v^*(t) = 1 \end{cases}$ for $0 \le t \le T$ leads to $P^f = (x_f,0)$ by the appropriate choice of ν_2. For different values of P^o, there are different values of ν_2. One such P^o for which ν_2 is such that $\tau_1 < \bar{\tau}_1 < \tilde{\tau}_1^o$ is shown in Figure 4. Since $\frac{dS_v}{d\tau} > 0$, we must have $v^*(\tau) = 0$ for $\tau > \bar{\tau}_1$ so that working backwards yields that the extremal labelled (2) passes through P^o. Thus, the figure suggests (also look at extremals emanating from $\bar{P}^o$) that there are several ways to reach the terminal manifold from a subset of the state space. The optimality of extremal trajectories has not been determined for this special case of multiple extremals. We will report our results to date with the final determination for the complete solution to (50) in Case (I2): $v_X < \frac{A}{a}$ and $v_Y \ge \frac{B}{b}$ being deferred to the future. In other words, the solution has not been completely determined in the shaded region of Figure 4.

In considering Figure 4, let us consider the case in which we have

$$\begin{cases} u^*(t) = 0 \\ v^*(t) = \bar{v}(t) \end{cases} \quad \text{for} \quad 0 \le t \le T. \tag{133}$$

---

*The reader is cautioned that Figure 4 represents a two-dimensional (i.e. t and y) illustration of the <u>projection</u> of extremals onto a plane parallel to the plane $x = 0$ in the three dimensional state space (i.e. t, x, and y space). For notational convenience, we are not being precise about which entities are in the (t,x,y)-space and which are projections onto the above noted subspace. Thus, C^o denotes both a figure in (t,x,y)-space and also its projection on (t,y)-space. Similar remarks apply to Figure 5.



Note: <u>Strategic Variables</u>	
X:	u
Y:	v

Figure 4. Dependence of Y Switching Time $\tilde{\tau}_1 = \tau_v$ Upon Force Levels for $bv_Y > B$ (<u>not drawn to scale</u>).

Then (133) may be combined with the state equations (50) and integrated to yield

$$x(t) = x_0 + rt - a\int_0^t \{1 - \bar{v}(s)\}y(s)ds, \tag{134}$$

and

$$y(t) = y_0 + st - b\int_0^t x(s)ds. \tag{135}$$

The above may be combined to yield the following (Volterra integral equation) for $y(t)$:

$$y(t) = y_0 + (s-bx_0)t - \frac{br}{2}t^2 + ab\int_0^t ds_1 \int_0^{s_1} \{1-\bar{v}(s_2)\}y(s_2)ds_2 \tag{136}$$

When $\bar{v}(s) = 0$ for $0 \le s \le t$, we obtain

$$y^{\bar{v}=0}(t) = y_0 + (s-bx_0)t - \frac{br}{2}t^2 + ab\int_0^t ds_1 \int_0^{s_1} y(s_2)ds_2, \tag{137}$$

and when $\bar{v}(s) = 1$ for $0 \le s \le t$, we obtain

$$y^{\bar{v}=1}(t) = y0 + (s-bx_0)t - \frac{br}{2}t^2. \tag{138}$$

It is also readily shown by direct integration of (50) with (133) when $\bar{v}(s) = 0$ for $0 \le s \le t$ that

$$y^{\bar{v}=0}(t) = \frac{1}{a}(ay_0-r)\cosh\sqrt{ab}\ t + \left(\frac{s-bx_0}{\sqrt{ab}}\right)\sinh\sqrt{ab}\ t + \frac{r}{a}. \tag{139}$$

Considering (137) and (138), it is clear that

$$y^{\bar{v}=0}(t) > y^{\bar{v}=1}(t) \quad \text{for} \quad t > 0, \tag{140}$$

provided that $y(t) \ge 0$ and that strict inequality holds for a finite

interval of time. Hence, whenever an (initial*) point lies on C^o (such as (B) = P^o in Figure 5), if Y uses the extremal strategy $v^*(t) = 0$ for $0 = T - \bar{\tau}_1 \le t \le T - \tau_1$ when $bv_Y > B$, the resulting extremal trajectory lies (strictly) above that corresponding to $v^*(t) = 1$ for $0 \le t \le T$. This situation is shown in Figure 5 (which is expanded detail of Figure 4) below. Thus, at point (B) = P^o (which lies on C^o) in Figure 5 Y can use an extremal strategy which leads either in the direction denoted as (3) or (4) (and results in extremals denoted as (b) and (c)). Moreover, if the trajectory began with $t = 0$ at, for example, point (A), then the extremal strategy $v^*(t) = 0$ for $0 \le t \le T - \bar{\tau}_1$ must have been used, since if extremal (c) is chosen, $S_v(\tau=\bar{\tau}_1) > 0$ and $\frac{dS_v}{d\tau} > 0$, and if extremal (b) is chosen, $S_v(\tau=\bar{\tau}_1) = 0$ and $\frac{dS_v}{d\tau} > 0$. Hence, C^o itself may be a transition surface (depending upon the choice of Y when C^o is first encountered). Furthermore, at (A) Y can choose either of two extremal directions denoted as (1) and (2) (and resulting in an extremal denoted as (a) or the one which leads to (B)). Thus, the problem apparently has multiple extremals (in the "triangular" region denoted as $\bar{P}^o\bar{P}Q$). It appears as though there may exist extremal trajectories (such as the one discussed above) at various points along which Y has a choice of alternative extremal strategies (i.e. a bifurcation occurs) that result in extremal trajectories denoted as (a), (b), and (c).

It has not been determined (by direct computation of the criterion functional) which extremal strategy is optimal for Y in this special set

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*The diagrams of battle trajectories in this report are sketched for a forward evolution of the battle (e.g. a curve of $y(t)$ for constant x_0). Isaacs [7] considers a backward evolution (e.g. a curve of $y(\tau)$ for constant $x_f = x(t=T)$). Neither approach is sufficient to determine optimal strategies. One must either invoke sufficient conditions for optimality or use the solution methodology referenced in [19] (see also [18], [21]). We hope to do this in the future.

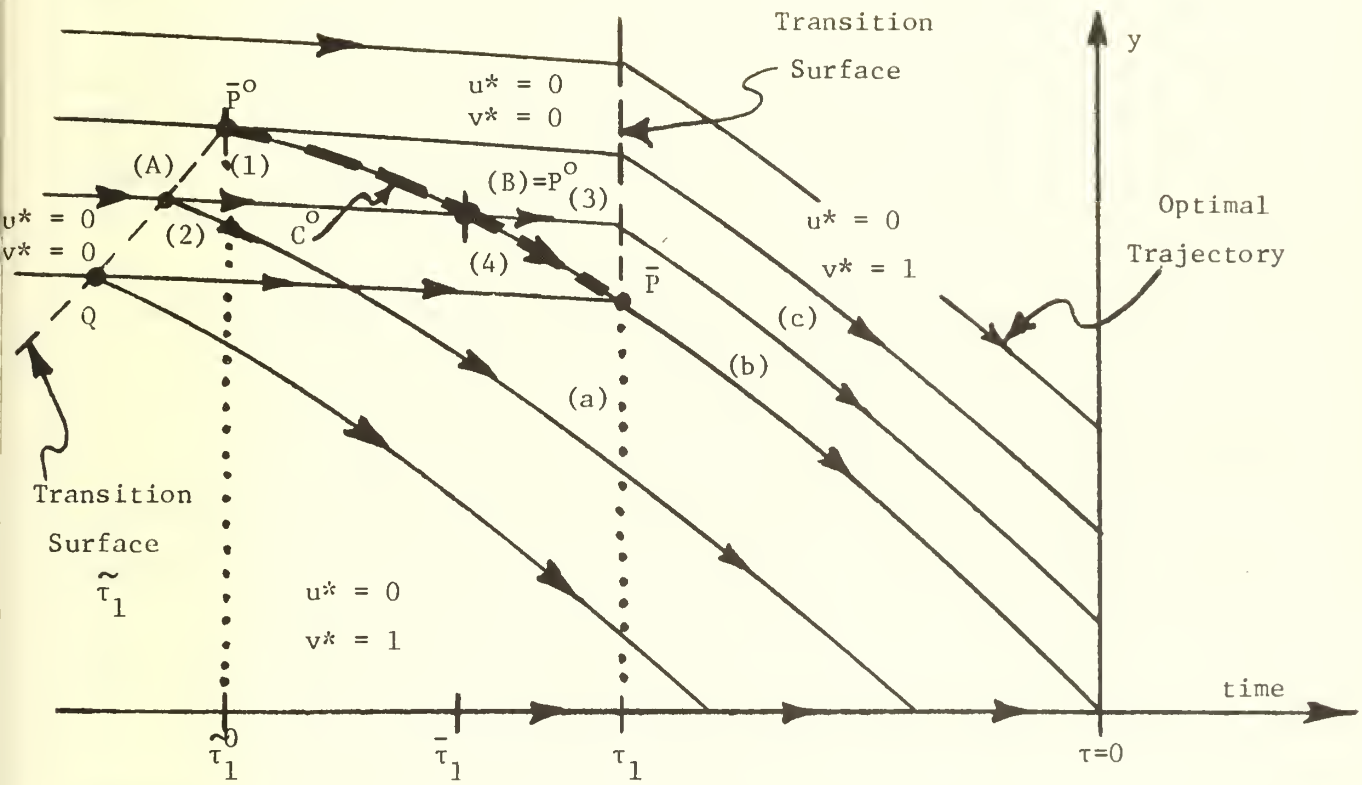

Figure 5. Illustration of Multiple Extremal Strategies v^* for Y when $bv_Y > B$ (not drawn to scale).

of circumstances. However, considering well-known sufficient conditions of global optimality based on convexity for optimal control problems, the author conjectures that all extremal strategies are optimal for Y and lead to the same return. If this is true, it means that there is an "indifference" set in Y's optimal strategy space such that he can vary the survivability of his aircraft at will without changing the value of his criterion functional. It should be noted that this occurs for $v_Y > \frac{B}{b}$, i.e. the Y aircraft (speaking somewhat imprecisely now) are valued greater than the return obtained from gound support missions at the end of the war.

Case (II1): $v_X \geq \frac{A}{a}$ and $v_Y < \frac{B}{b}$.

This case is similar to Case (I2) ($v_X < \frac{A}{a}$ and $v_Y \geq \frac{B}{b}$) with the roles of X and Y being interchanged.

Case (II2): $v_X \geq \frac{A}{a}$ and $v_Y \geq \frac{B}{b}$.

By Proposition 1 an optimal strategy for Y can lead to $x^*(T) = 0$ so that by (64) we have

$$S_u(\tau=0) = (av_X - A) - a\nu_1 \quad \text{where} \quad \nu_1 \begin{cases} = 0 & \text{for } x(T) > 0, \\ \geq 0 & \text{for } x(T) = 0. \end{cases}$$

There are two cases to be considered:

(A) $x(t=T) > 0$,

(B) $x(t=T) = 0$.

Similarly, by Proposition 3 an optimal strategy for X can lead to $y^*(T) = 0$ so that by (64) we again have (95). There are two further cases to be considered:

(a) $y(t=T) > 0$,

(b) $y(t=T) = 0$.

The tedious treatment of the above four cases (i.e. (Aa), (Ab), (Ba), and (Bb) where (Aa) denotes that we have $x(t=T) > 0$ and $y(t=T) > 0$ is similar to that done in Case (I2) ($v_X < \frac{A}{a}$ and $v_Y \geq \frac{B}{b}$) and therefore the details will be omitted. It may be shown that in all cases (i.e. (Aa), (Ab), (Ba), and (Bb)) we must have

$$S_u(\tau=0) \geq 0 \quad \text{and} \quad S_v(\tau=0) \geq 0, \tag{142}$$

so that, for example, we have $0 \leq \nu_1 \leq \left(\frac{av_X - A}{a}\right)$ when $x^*(t=T) = 0$. Considering (53), (54), (62), (63), and (142), arguments similar to those used above readily yield

$$u^*(\tau) = \begin{cases} 0 & \text{for } y(\tau) > 0 \\ 1 - \frac{s}{bx} & \text{for } y(\tau) = 0 \end{cases}$$
$$\text{for all } \tau \in [0,T]. \tag{143}$$
$$v^*(\tau) = \begin{cases} 0 & \text{for } x(\tau) > 0 \\ 1 - \frac{r}{ay} & \text{for } x(\tau) = 0 \end{cases}$$

Extremals are unique and therefore optimal.

5.2. Optimal Air-War Strategies.

In this section we will summarize the results of the previous section. In summarizing the optimal air-war strategies for X and Y, we must consider the four cases (i.e. (I1), (I2), (II1), and (II2)) delineated above.

Case (I1): $av_X < A$ and $bv_Y < B$.

We make the nonrestrictive assumption that $\left(\frac{A - av_X}{aB}\right) < \left(\frac{B - bv_Y}{bA}\right)$.

(1) If $T \leq \frac{A - av_X}{aB} = \tau_1$,

$$\begin{cases} U^*(t,x,y) = 1 \\ V^*(t,x,y) = 1 \end{cases} \quad \text{for } 0 \leq t \leq T.$$

(2) If $\frac{A-av_X}{aB} < T \le \frac{1}{aB}\left\{\sqrt{A^2+2aAB\left\{\left(\frac{B-bv_Y}{bA}\right) - \left(\frac{A-av_X}{aB}\right)\right\}} - av_X\right\}$,

$$\begin{cases} U^*(t,x,y) = 1 \\ V^*(t,x,y) = \begin{cases} 0 & \text{for } x > 0 \\ 1 - \frac{r}{ay} & \text{for } x = 0 \end{cases} \end{cases} \quad \text{for } 0 \le t \le T - \left(\frac{A-av_X}{aB}\right),$$

and

$$\begin{cases} U^*(t,x,y) = 1 \\ V^*(t,x,y) = 1 \end{cases} \quad \text{for } T - \left(\frac{A-av_X}{aB}\right) < t \le T.$$

(3) If $T > \frac{1}{aB}\left\{\sqrt{A^2+2aAB\left\{\left(\frac{B-bv_Y}{bA}\right) - \left(\frac{A-av_X}{aB}\right)\right\}} - av_X\right\} = \tau_2$,

$$\begin{cases} U^*(t,x,y) = \begin{cases} 0 & \text{for } y > 0 \\ 1 - \frac{s}{bx} & \text{for } y = 0 \end{cases} \\ V^*(t,x,y) = \begin{cases} 0 & \text{for } x > 0 \\ 1 - \frac{r}{ay} & \text{for } x = 0 \end{cases} \end{cases} \quad \text{for } 0 \le t \le T - \tilde{\tau}_2,$$

where

(a) $\tilde{\tau}_2 = \tau_2$ for $x(t=T-\tau_1) > 0$,

(b) $\tilde{\tau}_2 = \tau_e^X + \frac{1}{aB}\left\{\sqrt{A +2aAB\left\{\left(\frac{B-bv_Y}{bA}\right) - \tau_e^X\right\}} - A\right.$ for $x(t=T-\tau_1) = 0$,

where τ_e^X is largest value of τ such that $x(t=T-\tau_e^X) = 0$ and $\frac{A-av_X}{aB} = \tau_1 \le \tau_e^X \le \tau_2' = \frac{B-bv_Y}{bA}$,

(c) $\tilde{\tau}_2 = \tilde{\tau}_2'$ for $x(t=T-\tau_1) = 0$ when $\tau_e^X > \tilde{\tau}_2'$,

also

$$\begin{cases} U^*(t,x,y) = 1 \\ V^*(t,x,y) = \begin{cases} 0 & \text{for } x > 0 \\ 1 - \frac{r}{ay} & \text{for } x = 0 \end{cases} \end{cases} \quad \text{for } T - \tilde{\tau}_2 < t \le T - \left(\frac{A-av_X}{aB}\right),$$

and

$$\begin{cases} U^*(t,x,y) = 1 \\ V^*(t,x,y) = 1 \end{cases} \quad \text{for} \quad T - \left(\frac{A-av_X}{aB}\right) < t \leq T.$$

<u>Case (I2): $av_X < A$ and $bv_Y \geq B$.</u>

(1) If $T \leq \frac{1}{A}\left\{\sqrt{v_Y^2 + \frac{2A}{ab}(A-av_X)} = v_Y\right\} = \tau_1$,

$$\begin{cases} U^*(t,x,y) = \begin{cases} 0 & \text{for} \quad y > 0 \\ 1 - \dfrac{s}{bx} & \text{for} \quad y = 0 \end{cases} \\ V^*(t,x,y) = 1 \end{cases} \quad \text{for} \quad 0 \leq t \leq T.$$

(2)* If $T > \frac{1}{A}\left\{\sqrt{v_Y^2 + \frac{2A}{ab}(A-av_X)} - v_Y\right\} = \tau_1$,

$$\begin{cases} U^*(t,x,y) = \begin{cases} 0 & \text{for} \quad y > 0 \\ 1 - \dfrac{s}{bx} & \text{for} \quad y = 0 \end{cases} \\ V^*(t,x,y) = \begin{cases} 0 & \text{for} \quad x > 0 \\ 1 - \dfrac{r}{ay} & \text{for} \quad x = 0 \end{cases} \end{cases} \quad \text{for} \quad 0 \leq t \leq T - \tilde{\tau}_1,$$

where

(a) $\tilde{\tau}_1 = \tau_1$ for $y(t=T) > 0$,

(b) $\tilde{\tau}_1 = \tau_e^Y + \frac{1}{bA}\left\{\sqrt{B + 2bAB\left\{\left(\frac{A-av_X}{aB}\right) - \tau_e^Y\right\}} - B\right\}$ for $y(t=T) = 0$,

where τ_e^Y is largest value of τ such that $y(t=T-\tau_e^Y) = 0$ and $0 \leq \tau_e^Y \leq \tilde{\tau}_1' = \dfrac{A-av_X}{aB}$,

(c) $\tilde{\tau}_1 = \tilde{\tau}_1'$ for $y(t=T) = 0$ when $\tau_e^Y > \tilde{\tau}_1'$,

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*A final determination of the optimality of the extremal strategies still remains to be done in this case. Furthermore, the extremal strategy for Y is not unique. For convenience only one of two possibilities is given here.

and

$$\begin{cases} U^*(t,x,y) = \begin{cases} 0 & \text{for} \quad y > 0 \\ 1 - \dfrac{s}{bx} & \text{for} \quad y = 0 \end{cases} & \text{for} \quad T - \tilde{\tau}_1 < t \leq T. \\ V^*(t,x,y) = 1 & \end{cases}$$

Case (I11): $av_X \geq A$ and $bv_Y < B$.

(1) If $T \quad \frac{1}{B}\left\{\sqrt{v_X^2 + \frac{2B}{ab}(B - bv_Y)} - v_X\right\} = \tau_1$,

$$\begin{cases} U^*(t,x,y) = 1 & \\ V^*(t,x,y) = \begin{cases} 0 & \text{for} \quad x > 0 \\ 1 - \dfrac{r}{ay} & \text{for} \quad x = 0 \end{cases} & \text{for} \quad 0 \quad t \quad T. \end{cases}$$

(2)* If $T > \frac{1}{B}\left\{\sqrt{v_X^2 + \frac{2B}{ab}(B - bv_Y)} - v_X\right\} = \tau_1$,

$$\begin{cases} U^*(t,x,y) = \begin{cases} 0 & \text{for} \quad y > 0 \\ 1 - \dfrac{s}{bx} & \text{for} \quad y = 0 \end{cases} & \\ & \text{for} \quad 0 \quad t \quad T - \tilde{\tau}_1, \\ V^*(t,x,y) = \begin{cases} 0 & \text{for} \quad x > 0 \\ 1 - \dfrac{r}{ay} & \text{for} \quad x = 0 \end{cases} & \end{cases}$$

where

(a) $\tilde{\tau}_1 = \tau_1$ for $x(t=T) > 0$,

(b) $\tilde{\tau}_1 = \tau_e^X + \frac{1}{aB}\left\{\sqrt{A^2 + 2aAB\left\{\left(\frac{B - bv_Y}{bA}\right) - \tau_e^X\right\}} - A\right\}$ for $x(t=T) = 0$,

where τ_e^X is largest value of τ such that $x(t=T=\tau_e^X) = 0$

and $0 \leq \tau_e^X \leq \tilde{\tau}_1' = \dfrac{B - bv_Y}{bA}$,

(c) $\tilde{\tau}_1 = \tilde{\tau}_1'$ for $x(t=T) = 0$ when $\tau_e^X > \tilde{\tau}_1'$,

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*A final determination of the optimality of the extremal strategies still remains to be done in this case. See also other comments in footnote for Case (I2).

and

$$\begin{cases} U^*(t,x,y) = 1 \\ V^*(t,x,y) = \begin{cases} 0 & \text{for} \quad x \quad 0 \\ 1 - \dfrac{r}{ay} & \text{for} \quad z = 0 \end{cases} \end{cases} \qquad \text{for} \quad T < \tilde{\tau}_1 < t \leq T.$$

Case (II2): $av_X \quad A$ and $bv_Y \quad B$.

$$\begin{cases} U^*(t,x,y) = \begin{cases} 0 & \text{for} \quad y > 0 \\ 1 - \dfrac{s}{bx} & \text{for} \quad y = 0 \end{cases} \\ V^*(t,x,y) = \begin{cases} 0 & \text{for} \quad x > 0 \\ 1 - \dfrac{r}{ay} & \text{for} \quad x = 0 \end{cases} \end{cases} \qquad \text{for} \quad 0 \leq t \leq T.$$

6. Some Results for Case of Linearly-Decreasing-with-Time Returns from Ground Support.

One criticism of the constant coefficient version (50) of the generalization (1) of the tactical air-war game is that the criterion functional does not adequately reflect timeliness of ground support. Intuitively, we would expect that support of the ground forces is "worth more" in the early stages of the ground war than later on. Thus, we are led to consider time-dependent "returns" from ground support that decrease over time. Two cases of time-dependent returns from ground support that are of particular interest are*

(1) linearly-decreasing-with-time returns from ground support,

(2) exponentially-decreasing-with-time returns from ground support.

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*Interest in these cases was confirmed in informal discussions with Maj Ron Kronz, USAF. Such facets have been apparently explored in numerous runs of TAC CONTENDER.

Let us consider the case of (1) in which the Lanchester attrition-rate coefficients for counter-air operations are constants and the returns from ground support are time dependent (decreasing over time). Thus, we will consider the case in which the following hold:

$a(t) = \bar{a} = \text{constant},$

$b(t) = \bar{b} = \text{constant},$

$A(t)$ is given (decreasing) function of time,

$B(t)$ is given (decreasing) function of time.

For notational convenience we will again denote $\bar{a}$ as a and $\bar{b}$ as b. It is then convenient to re-state the problem as follows:

$$\underset{U}{\text{maximize}}\ \underset{V}{\text{minimize}}\ \{v_X x(t_f) - v_Y y(t_f) + \int_0^{t_f} [B(t)ux - A(t)vy]dt\},$$

with stopping rule: $t_f - T = 0$,

subject to:
(air-battle dynamics)

$$\frac{dx}{dt} = r - (1-v)ay, \tag{144}$$

$$\frac{dy}{dt} = s - (1-u)bx,$$

with initial conditions $x(t=0) = x_0$ and $y(t=0) = y_0$,

and

$x,y \geq 0$ (State Variable Inequality Constraints),

$0 \leq u,v \leq 1$ (Strategic Variable Inequality Constraints),

where a and b are given constants and $A(t)$ and $B(t)$ are given functions.

In this section we will consider the case in which both $A(t)$ and $B(t)$ are linearly decreasing functions of time for $0 \leq t \leq T$. The situation for $A(t)$ is shown in Figure 6. We have then that

$$A(t) = A_0 - t\left(\frac{A_0 - A_T}{T}\right), \tag{145}$$

and

$$B(t) = B_0 - t\left(\frac{B_0 - B_T}{T}\right), \tag{146}$$

where $A_0 > A_T$ and $B_0 > B_T$.

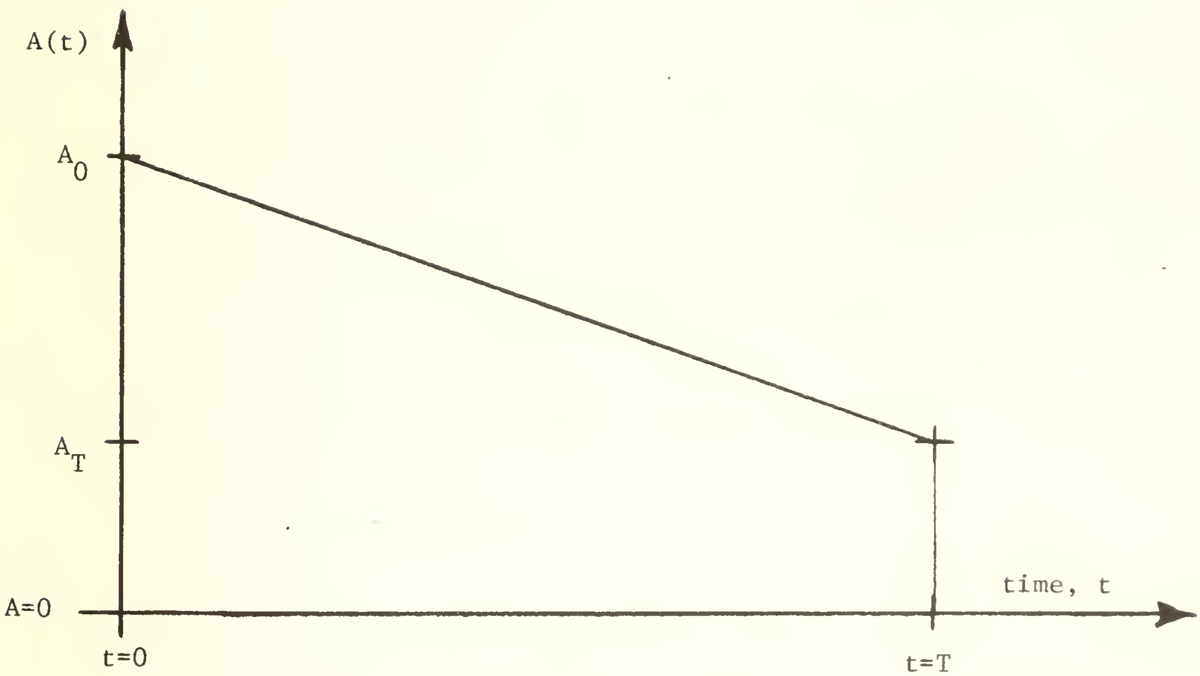

Figure 6. Linearly-Decreasing-with-Time Returns from Ground Support $A(t)$.

We will now review the necessary conditions of optimality that we have developed for (144) in Section 3 before partially synthesizing the extremal strategies in one of four cases. We hope to give a more thorough study of this important problem in the future. Let us recall the <u>switching functions</u> in <u>backwards time</u> $\tau = T - t$

$$S_u(\tau) = b(-q(\tau)) - \hat{B}(\tau), \tag{147}$$

$$S_v(\tau) = ap(\tau) - \hat{A}(\tau), \tag{148}$$

where

$$\hat{A}(\tau) = A(T-\tau) = A_T + \tau\left(\frac{A_0-A_T}{T}\right) \tag{149}$$

and

$$\hat{B}(\tau) = B(T-\tau) = B_T + \tau\left(\frac{B_0-B_T}{T}\right) \tag{15}$$

Again, extremal strategies are given by (53) and (54) for $x,y > 0$. The differential equations for the switching functions are given by

$$\frac{dS_u}{d\tau} = \begin{cases} b\{\hat{A}(\tau) + (10v^*)S_v(\tau)\} - \dfrac{d\hat{B}}{d\tau} & \text{for } y > 0, \\ 0 \quad \text{for } y = 0 & \text{(for a finite interval of time),} \end{cases} \tag{151}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} a\{\hat{B}(\tau) + (1-u^*)S_u(\tau)\} - \dfrac{d\hat{A}}{d\tau} & \text{for } x > 0, \\ 0 \quad \text{for } x = 0 & \text{(for a finite interval of time),} \end{cases} \tag{152}$$

with initial conditions

$$S_u(\tau=0) = (bv_Y-B_T) - bv_2 \quad \text{and} \quad S_v(\tau=0) = (av_X-A_T) - av_1, \tag{153}$$

where ν_1 and ν_2 must satisfy (60) and (61), respectively. It is also convenient to write (151) and (152) as

$$\frac{dS_u}{d\tau} = \begin{cases} b\{A_T + \tau\left(\frac{A_0 - A_T}{T}\right) + (1-v^*)S_v(\tau) & - \left(\frac{B_0 - B_T}{T}\right) \quad \text{for} \quad y > 0, \\ 0 \quad \text{for} \quad y = 0 \quad \text{(for a finite interval of time)}, \end{cases} \tag{154}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} a\{B_T + \tau\left(\frac{B_0 - B_T}{T}\right) + (1-u^*)S_u(\tau)\} - \left(\frac{A_0 - A_T}{T}\right) \quad \text{for} \quad x > 0, \\ 0 \quad \text{for} \quad x = 0 \quad \text{(for a finite interval of time)}. \end{cases} \tag{155}$$

It is readily shown that singular solutions are impossible. For example, when $x,y > 0$ and $0 < v^* < 1$, we have

$$\frac{d^2S_u}{d\tau^2} = b\left(\frac{A_0 - A_T}{T}\right) > 0,$$

and when v^* is constant (i.e. the other cases in which $v^* = 0$ or 1), we have

$$\frac{d^2S_u}{d\tau^2} = b\{v^*\left(\frac{A_0 - A_T}{T}\right) + (1-v^*)a[B_T + \tau\left(\frac{B_0 - B_T}{T}\right) + (1-u^*)S_u(\tau)]\}.$$

Observing that $(1-u^*)S_u(\tau) \geq 0$, it is clear that $\frac{d^2S_u}{d\tau^2} > 0$ for all τ in the latter case, since it is a convex combination of positive quantities.

On a constrained subarc on which $x(t) = 0$ for $t_e^X \leq t \leq t^X$ we have shown the following to be necessary conditions for optimality:

$$v^*(t) = 1 - \frac{r}{ay(t)} \qquad \text{for} \quad t_e^X < t < t_\ell^X, \tag{156}$$

$$r < ay(t) \qquad \text{for} \quad t_e^X \leq t \leq t_\ell^X \tag{157}$$

$$p(t) = \frac{A(t)}{A} \qquad \text{for} \quad t_e^X \leq t \leq t_\ell^X, \tag{158}$$

and

$$B(t)u^* + (1-u^*)b(-q(t)) \geq \frac{1}{a}\left(\frac{A_0 - A_T}{T}\right). \tag{159}$$

Other convenient equivalent forms of (159) are

$$a\{B(t) + (1-u^*)S_u(t)\} \geq \left(\frac{A_0 - A_T}{T}\right), \tag{160}$$

or

$$\text{maximum}\{B(t),\ b(pq(t))\} \geq \frac{1}{a}\left(\frac{A_0 - A_T}{T}\right). \tag{161}$$

It is to be noted that (159) is no longer automatically satisfied on constrained subarcs. Similar conditions hold on a constrained subarc on which $y(t) = 0$ for a finite interval of time.

Concerning extremal end states, we have developed the following.

PROPOSITION 5: If an optimal strategy for Y is to result in $x^*(T) = 0$, then it is necessary that

$$v_X \geq \frac{A_T}{a}.$$

PROPOSITION 6: If $v_X < \frac{A_T}{a}$, then an optimal strategy for Y leads to $x^*(T) > 0$.

PROPOSITION 7: If an optimal strategy for X is to result in $y^*(T) = 0$, then it is necessary that

$$v_Y \geq \frac{B_T}{b}.$$

PROPOSITION 8: If $v_Y < \frac{B_T}{b}$, then an optimal strategy for X leads to $y^*(T) > 0$.

In synthesizing an extremal strategic-variable pair there are four cases to be considered:

(I) $v_X < \frac{A_T}{a}$

(1) $v_Y < \frac{B_T}{b}$

(2) $v_Y \geq \frac{B_T}{b}$

(II) $v_X \leq \frac{A_T}{b}$

(1) $v_Y < \frac{B_T}{b}$

(2) $v_Y \geq \frac{B_T}{b}$.

$$\frac{dS_u}{d\tau} = \begin{cases} b\{A_T + \tau\left(\frac{A_0 - A_T}{T}\right) + (1-v^*)S_v(\tau) & - \left(\frac{B_0 - B_T}{T}\right) \quad \text{for } y > 0, \\ 0 \quad \text{for } y = 0 \quad \text{(for a finite interval of time),} \end{cases} \tag{154}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} a\{B_T + \tau\left(\frac{B_0 - B_T}{T}\right) + (1-u^*)S_u(\tau)\} - \left(\frac{A_0 - A_T}{T}\right) & \text{for } x > 0, \\ 0 \quad \text{for } x = 0 \quad \text{(for a finite interval of time).} \end{cases} \tag{155}$$

It is readily shown that singular solutions are impossible. For example, when $x,y > 0$ and $0 < v^* < 1$, we have

$$\frac{d^2S_u}{d\tau^2} = b\left(\frac{A_0 - A_T}{T}\right) > 0,$$

and when v^* is constant (i.e. the other cases in which $v^* = 0$ or 1), we have

$$\frac{d^2S_u}{d\tau^2} = b\{v^*\left(\frac{A_0 - A_T}{T}\right) + (1-v^*)a[B_T + \tau\left(\frac{B_0 - B_T}{T}\right) + (1-u^*)S_u(\tau)]\}.$$

Observing that $(1-u^*)S_u(\tau) \geq 0$, it is clear that $\frac{d^2S_u}{d\tau^2} > 0$ for all τ in the latter case, since it is a convex combination of positive quantities.

On a constrained subarc on which $x(t) = 0$ for $t_e^X \leq t \leq t^X$ we have shown the following to be necessary conditions for optimality:

$$v^*(t) = 1 - \frac{r}{ay(t)} \qquad \text{for } t_e^X < t < t_\ell^X, \tag{156}$$

$$r < ay(t) \qquad \text{for } t_e^X \leq t \leq t_\ell^X \tag{157}$$

$$p(t) = \frac{A(t)}{A} \qquad \text{for } t_e^X \leq t \leq t_\ell^X, \tag{158}$$

and

$$B(t)u^* + (1-u^*)b(-q(t)) \geq \frac{1}{a}\left(\frac{A_0 - A_T}{T}\right). \tag{159}$$

Other convenient equivalent forms of (159) are

$$a\{B(t) + (1-u^*)S_u(t)\} \geq \left(\frac{A_0 - A_T}{T}\right), \tag{160}$$

or

$$\text{maximum}\{B(t),\ b(pq(t))\} \geq \frac{1}{a}\left(\frac{A_0 - A_T}{T}\right). \tag{161}$$

It is to be noted that (159) is no longer automatically satisfied on constrained subarcs. Similar conditions hold on a constrained subarc on which $y(t) = 0$ for a finite interval of time.

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PROPOSITION 5: If an optimal strategy for Y is to result in $x^*(T) = 0$, then it is necessary that

$$v_X \geq \frac{A_T}{a}.$$

PROPOSITION 6: If $v_X < \frac{A_T}{a}$, then an optimal strategy for Y leads to $x^*(T) > 0$.

PROPOSITION 7: If an optimal strategy for X is to result in $y^*(T) = 0$, then it is necessary that

$$v_Y \geq \frac{B_T}{b}.$$

PROPOSITION 8: If $v_Y < \frac{B_T}{b}$, then an optimal strategy for X leads to $y^*(T) > 0$.

In synthesizing an extremal strategic-variable pair there are four cases to be considered:

(I) $v_X < \frac{A_T}{a}$

(1) $v_Y < \frac{B_T}{b}$

(2) $v_Y \geq \frac{B_T}{b}$

(II) $v_X \leq \frac{A_T}{b}$

(1) $v_Y < \frac{B_T}{b}$

(2) $v_Y \geq \frac{B_T}{b}$.

We will give some initial results in the first of the above cases.

In Case (ii): $v_X < \frac{A_T}{a}$ and $v_Y < \frac{B_T}{b}$, we have by Propositions 6 and 8 that $x^*(T) > 0$ and $y^*(T) > 0$ so that by (153) we have

$$S_u(\tau=0) = bv_Y - B_T < 0 \quad \text{and} \quad S_v(\tau=0) = av_X - A_T < 0. \tag{162}$$

By the usual arguments, it is readily seen that

$$\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases} \quad \text{for} \quad \tau \in [0,\tau_1], \tag{163}$$

and $x(\tau), y(\tau) > 0$ for $\tau \in [0,\tau_1)$. Using (162) and (163), the differential equations for the switching functions may be integrated to yield for $0 \le \tau \le \tau_1$

$$S_u(\tau) = \frac{b}{2}\left(\frac{A_0-A_T}{T}\right)\tau^2 + \{bA_T - \left(\frac{B_0-B_T}{T}\right)\}\tau + bv_Y - B_T, \tag{164}$$

and

$$S_v(\tau) = \frac{a}{2}\left(\frac{B_0-B_T}{T}\right)\tau^2 + \{aB_T - \left(\frac{A_0-A_T}{T}\right)\}\tau + av_X - A_T. \tag{165}$$

Let τ_u denote the (only) positive root of $S_u(\tau=\tau_u) = 0$ and similarly for τ_v. Then we have

$$\tau_u = -\left\{T\left(\frac{A_T}{A_0-A_T}\right) - \frac{1}{b}\left(\frac{B_0-B_T}{A_0-A_T}\right)\right\} + \sqrt{\left\{T\left(\frac{A_T}{A_0-A_T}\right) - \frac{1}{b}\left(\frac{B_0-B_T}{A_0-A_T}\right)\right\}^2 + \frac{2T}{b}\left(\frac{B_T-bv_Y}{A_0-A_T}\right)}, \tag{166}$$

and

$$\tau_v = -\left\{T\left(\frac{B_T}{B_0-B_T}\right) - \frac{1}{a}\left(\frac{A_0-A_T}{B_0-B_T}\right)\right\} + \sqrt{\left\{T\left(\frac{B_T}{B_0-B_T}\right) - \frac{1}{a}\left(\frac{A_0-A_T}{B_0-B_T}\right)\right\}^2 + \frac{2T}{a}\left(\frac{A_T-av_X}{B_0-B_T}\right)}. \tag{167}$$

The backwards switching time τ_1 is then given by

$$\tau_1 = \text{minimum}\ (\tau_u, \tau_v) \le T. \tag{168}$$

It should be noted that in order for the switch to actually occur we require that $\tau_1 \le T$. Omitting the rather tedious details, it is a straightforward matter to prove the following theorem:

THEOREM 3: In order that $\tau_u \in [0,T)$ for $B_T > bv_Y$,

(1) it is necessary that $\left(\frac{B_0-B_T}{T}\right) < bA_0$,

(2) it is necessary and sufficient that

(a) $\left(\frac{B_0-B_T}{T}\right) < bA_0$,

and

(b) $\left\{\frac{B_0-bv_Y}{b\left(\frac{A_0-A_T}{2}\right)}\right\} < T$,

A similar theorem is readily proven concerning τ_v. Thus, we see that in order for a switch in extremal strategies from (163) to occur additional conditions are required. This should be contrasted with the constant coefficient case considered in Section 5 in which for the analogous situation (i.e. Case (I1)). A switch in extremal strategies from (68) was guaranteed if the duration of the battle was long enough (i.e. $T > \frac{A-av_X}{aB}$). As an immediate corollary to Theorem 3, we have

COROLLARY 3.1: If $\left(\frac{A_0-A_T}{T}\right) \ge aB_0$ and $\left(\frac{B_0-B_T}{T}\right) \ge bA_0$, then the solution to (144) with $A(t)$ and $B(t)$ given by (145) and (146) is given by

$$\begin{cases} u^*(t) = 1 \\ v^*(t) = 1 \end{cases} \quad \text{for } 0 \le t \le T.$$

It should be noted that condition (2b) of Theorem 3 may be written as

$$\frac{B(t=0) - bv_Y}{b\bar{A}} < T, \tag{169}$$

where

$$\bar{A} = \frac{1}{T} \int_0^T A(t)dt.$$

Shortness of time precludes further results for this problem. The author hopes to give further details in the future.

7. <u>Some Results for Case of Exponentially-Decreasing-with-Time Returns from Ground Support</u>.

In this section we will briefly consider the case in which both $A(t)$ and $B(t)$ are exponentially decreasing functions of time for $0 \leq t \leq T$. We have then that

$$A(t) = A_0 \, e^{-\alpha t}, \tag{170}$$

and

$$B(t) = B_0 \, e^{-\beta t}, \tag{171}$$

where $\alpha, \beta > 0$.

We will now review the review the necessary conditions of optimality that we have developed for (144) in Section (3) before partially synthesizing the extremal strategies in one of four cases. Again, we hope to give a more thorough study of this important problem in the future. As before, the extremal strategies are given by (53) and (54) for $x,y > 0$. The differential equations for the switching functions are given by

$$\frac{dS_u}{d\tau} = \begin{cases} b(1-v^*)S_v(\tau) + (bA_0e^{-\alpha T})e^{\alpha\tau} - (\beta B_0e^{-\beta T})e^{\beta\tau} & \text{for } y > 0, \\ 0 \quad \text{for } y = 0 & \text{(for a finite interval of time),} \end{cases} \tag{172}$$

and

$$\frac{dS_v}{d\tau} = \begin{cases} a(1-u^*)S_u(\tau) + (aB_0e^{-\beta T})e^{\beta\tau} - (\alpha A_0e^{-\alpha T})e^{\alpha\tau} & \text{for } x > 0, \\ 0 \quad \text{for } x = 0 & \text{(for a finite interval of time),} \end{cases} \tag{173}$$

with initial conditions

$$S_u(\tau=0) = (bv_Y - B_0e^{-\beta T}) - b\nu_2 \quad \text{and} \quad S_v(\tau=0) = (av_X - A_0e^{-\alpha T}) - a\nu_1, \tag{174}$$

where ν_1 and ν_2 must satisfy (60) and (61), respectively.

On a constrained subarc on which $x(t) = 0$ for $t_e^X \le t \le t^X$, we have shown that (156) through (158) are necessary conditions for optimality and also

$$B(t)u^* + (1-u^*)b(-q(t)) \ge \frac{A_0}{a} e^{-\alpha t}. \tag{175}$$

Other convenient equivalent forms of (175) are

$$a\{B(t) + (1-u^*)S_u(t)\} \ge \alpha A_0 e^{-\alpha t}, \tag{176}$$

or

$$\text{maximum } \{B(t), b(-q(t))\} \ge \frac{\alpha A_0}{a} e^{-\alpha t}. \tag{177}$$

As in the previous case, it is to be noted that (175) is no longer automatically satisfied on constrained subarcs. Similar conditions hold on a constrained subarc on which $y(t) = 0$ for a finite interval of time.

Concerning extremal end states, we have developed the following.

> PROPOSITION 9: If an optimal strategy for Y is to result in $x^*(T) = 0$, then it is necessary that
>
> $$v_X \ge \frac{A_0e^{-\alpha T}}{a} .$$

PROPOSITION 10: If $v_X < \frac{A_0 e^{-\alpha T}}{a}$, then an optimal strategy for strategy for Y leads to $x^*(T) > 0$.

PROPOSITION 11: If an optimal strategy for X is to result in $y^*(T) = 0$, then it is necessary that
$$v_Y \geq \frac{B_0 e^{-\beta T}}{b} .$$

PROPOSITION 12: If $v_Y < \frac{B_0 e^{-\beta T}}{b}$, then an optimal strategy for X leads to $y^*(T) > 0$.

In synthesizing an extremal strategic-variable pair there are four cases to be considered:

(I) $v_X < \frac{A_0 e^{-\beta T}}{a}$

(1) $v_Y < \frac{B_0 e^{-\beta T}}{b}$

(2) $v_Y \geq \frac{B_0 e^{-\beta T}}{b}$

(II) $v_X \leq \frac{A_0 e^{-\alpha T}}{a}$

(1) $v_Y < \frac{B_0 e^{-\beta T}}{b}$

(2) $v_Y \geq \frac{B_0 e^{-\beta T}}{b}$.

We will now give some initial results in the first of the above cases.

In <u>Case (I1): $v_X < \frac{A_0 e^{-\alpha T}}{a}$ and $b_Y < \frac{B_0 e^{-\beta T}}{b}$</u>, we have by Propositions 10 and 12 that $x^*(T) > 0$ and $y^*(T) > 0$ so that by (174) we have

$$S_u(\tau=0) = bv_Y - B_0 e^{-\beta T} < 0 \quad \text{and} \quad S_v(\tau=0) = av_X - A_0 e^{-\alpha T} < 0. \qquad (178)$$

By the usual arguments, it is readily seen that

$$\begin{cases} u^*(\tau) = 1 \\ v^*(\tau) = 1 \end{cases} \quad \text{for} \quad \tau \in [0, \tau_1], \qquad (179)$$

and $x(\tau), y(\tau) > 0$ for $\tau \in [0,\tau_1)$. Using (178) and (179), the differential equations for the switching functions may be integrated to yield for $0 \le \tau \le \tau_1$

$$S_u(\tau) = \left(\frac{bA_0e^{-\alpha T}}{\alpha}\right) e^{\alpha\tau} - (B_0e^{-\beta T})e^{\beta\tau} + b\left(v_Y - \frac{A_0e^{-\alpha T}}{\alpha}\right), \tag{180}$$

and

$$S_v(\tau) = \left(\frac{aB_0e^{-\beta T}}{}\right) e^{\beta\tau} - (A_0e^{-\alpha T})e^{\alpha\tau} + a\left(v_X - \frac{B_0e^{-\beta T}}{\beta}\right). \tag{181}$$

Let τ_u denote the smallest positive root of $S_u(\tau=\tau_u) = 0$ and similarly for τ_v. Then the backwards switching time τ_1 is given by

$$\tau_1 = \text{maximum } (\tau_u,\tau_v) \le T. \tag{182}$$

We will now give some results for the following two cases:

(1) $\alpha = \beta$,

(2) $\beta > \alpha$.

In <u>Case (1): $\alpha = \beta$</u>, in order for τ_u to exist such that $S_u(\tau=\tau_u) = 0$ we must have

$$bA_0 > \beta B_0, \tag{183}$$

and then

$$\tau_u = \frac{1}{\alpha} \ln \left\{\frac{b(A_0-\beta e^{\beta T}v_Y)}{bA_0 - \beta B_0}\right\}. \tag{184}$$

Omitting the details for now, the following theorem may be proven:

THEOREM 4: In order that $\tau_u \varepsilon [0,T)$ for $\alpha = \beta$ and $B_0 e^{-\beta T} > bv_Y$,

(1) it is necessary that $\frac{dB}{dt}(t=0) < bA(t=0)$ (where $A(t)$ and $B(t)$ are given by (170) and (171))

(2) it is necessary and sufficient that

(a) $\frac{dB}{dt}(t=0) < bA(t=0)$,

and

(b) $\left\{\frac{B(t=0) - bv_Y}{b\bar{A}}\right\} < T$,

where

$$\bar{A} = \frac{1}{T}\int_0^T A(t)dt.$$

A similar theorem concerning τ_v may be proven. The reader should note that Theorems 3 and 4 are basically identical. Thus, there is great similarity (at least for the results so far obtained) between the cases of linearly-decreasing-with-time returns from ground support and exponentially-decreasing ones. As above, we see that in order for a switch in extremal strategies from (179) to occur additional conditions in addition to the war being of sufficient duration are required. This should be contrasted with the constant coefficient case (i.e. Case (I1)) considered in Section 5. As an immediate corollary to Theorem 4, we have

COROLLARY 4.1: For the case in which $\alpha = \beta$, if $\frac{dA}{dt}(t=0) \geq aB(t=0)$ and $\frac{dB}{dt}(t=0) \geq bA(t=0)$ then the solution to (144) with $A(t)$ and $B(t)$ given by (170) and (171) is given by

$$\left\{\begin{array}{l} u^*(t) = 1 \\ v^*(t) = 1 \end{array}\right. \quad \text{for } 0 \leq t \leq T.$$

In Case (2): $\beta > \alpha$, the following theorem may be shown to hold.

THEOREM 5: In order that $\tau_u \in [0,T)$ for $\beta > \alpha$ and $B(t=T) > bv_Y$, it is sufficient that

$$\text{(a)} \quad \frac{dB}{dt}(t=0) < bA(t=0),$$

and

$$\text{(b)} \quad \left\{\frac{B(t=0) - bv_Y}{b\bar{A}}\right\} < T.$$

Shortness of time precludes further results for this problem. The author hopes to give further details in the future.

8. Discussion.

In this appendix we have applied the theory of differential games (including the theory of state variable inequality constraints developed in Appendix A) to the study of optimal air-war strategies. We have considered a generalization of the tactical air-war game studied by Isaacs [7] and others [1], [4], [6], [16], [24]. We saw that when one considers salvage values for aircraft, time-dependent returns from ground support, and time-dependent Lanchester attrition-rate coefficients, previous solution methodology for the tactical air-war game can result in non-optimal strategies. For example, during the early phases of the air war, it may not be optimal to annihilate enemy aircraft whenever possible. We also saw that when there is a residual value for aircraft at the "end" of the war, optimal strategies may be appreciably different than those for the special case previously considered.

We developed a (fairly) complete solution to the generalization of the tactical air-war game for the case of constant coefficients. Some work

still remains to be done in one special case. We propose this to ONR as a future research task. We also partially developed extremal strategies in two cases of time-varying (decreasing) returns from ground support (both linearly-decreasing returns and also exponentially-decreasing ones). In these cases we have shown that optimal strategies for both sides may be to always concentrate on supporting the ground forces regardless of how long the conflict lasts. For the case in which $av_X < A(t=T)$ and $bv_Y < B(t=T)$, we showed that an additional condition had to be satisfied (namely, $\frac{dB}{dt}(t=0) < bA(t=0)$) in order for it to be possible for a switch in X's optimal strategy to occur. (It should be noted that this latter condition is always satisfied for the case of constant returns from ground support.)

The results of this appendix are not only of intrinsic interest but are also useful for checking the computational adequacy of Lagrange dynamic programming, which has been proposed as a compational method for time-sequential combat games [13], [14]. It should be pointed out that there may be severe computational difficulties in using Lagrange dynamic programming when the solution to a time-sequential combat game has multiple extremals (as is frequently the case (see [19])).

We think that an understanding of the parameters which affect optimal air-war strategies is essential. In particular, the effects on optimal strategies of considering salvage values for aircraft, time-dependent returns for ground support, and time-dependent Lanchester attrition-rate coefficients are not entirely understood at this time. We feel that an analytic approach like that employed here is very useful for discovering cause-effect relationships between optimal strategies and modelling assumptions. We have laid the foundations for future research in this area and recommend such future

study to ONR. Of particular importance, is to determine whether or not the basic model is adequate. A basic shortcoming of the model (1) is that it does not evaluate air-war tactics within the context of ground-war objectives. The model suboptimizes. In the next appendix we briefly consider a model that does evaluate air-war tactics within the context of ground-war objectives.

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APPENDIX E: Some Problems for Future Study.

1. Introduction.

In this appendix we will briefly describe several important problems for possible future research. These problems are all extensions of work reported both here in this report and elsewhere [9], [10].

As we have pointed out recently [11], when we began our research, the state-of-the-art of deterministic control theory (at least that for applications to operations research (see [12])) was not sufficient to allow routine solution of even the simplest Lanchester-type optimal control problem. Our research approach has been to consider a sequence of idealized models of increasing mathematical complexity and also increasing operational realism.

At this point in time, most of the simplest Lanchester-type tactical allocation structures have been fairly thoroughly studied. Some theoretical results in optimal control theory have resulted from this examination [12], [13]. To solve the problems at hand that we propose, we would build upon the knowledge gained by solving numerous particular examples.

The problems that we will elaborate upon below may be referred to as

(1) fire support for several ground units,

(2) incorporation of logistics considerations,

(3) effect of the form of the criterion functional in combat optimization problems.

All the above problems are quite important. The first is important because it is a closer approximation to the type of decision that a commander must make in allocating supporting fires to aid various friendly units than we have previously considered. The second is important because it incorporates logistics considerations into Lanchester-type equations of warfare. As far as Navy missions are concerned, this is very important, since sea-based

supply and re-supply is a principle Navy combat support mission. The third is important because it provides more insight into the relationship between the structure of optimal strategies and the nature of (system) objectives (here nonlinear).

As we first pointed out in [11], for the purposes of military analysis, it is convenient to consider that there are three essential parts of any dynamic combat optimization problem:

(a) the decision criteria (for both combatants),

(b) the model of conflict termination conditions (and/or unit breakpoints),

(c) the model of combat dynamics.

Our proposed problems (1) and (2) above fall into the category of research on the effects on the structure of optimal strategies of aspect (c), whereas (3) falls into the category of research on (a). It is this central analysis aspect of the combatants' decision criteria that the author believes is the weakest link in the chain of assumptions in current analyses. Pugh and Mayberry [4] have recently proposed methodology for this important topic of measures of effectiveness in dynamic combat optimization problems. However, they do not explore the consequences of various functional forms of the criterion functional. This is the objective of studying problem (3).

Finally, it should be pointed out that all the problems considered below are stated in their simplest form. This is because of our research objective of studying solution properties, an objective whose attainment is facilitated by the availability of complete, explicit analytic solutions. More realistic versions of these problems are readily formulated. However, mathematical tractability is sacrificed when this is done. Moreover, our results for such idealized problems provides guidance for developing numerical solution algorithms for more realistic campaign models.

2. Fire Support of Several Ground Units.

Consider combat between heterogeneous X and Y ground forces (infantry). Contact exists between these forces along the Forward Edge of the Battle Area (FEBA). The X ground forces are composed of two units, denoted as X_1 and X_2, and similarly for the Y ground forces. It is assumed that the X_1 forces are only in contact (i.e. combat) with the Y_1 forces and similarly for X_2. In each of the two "one-on-one" battles both combatants undergo a "square-law" attrition process (see [6]) from enemy (small arms) fire.

The problem facing the X commander is to determine the "best" distribution of the fire of his artillery, denoted as U, over the enemy units, i.e. Y_1 and Y_2. The artillery U fires at a constant rate into the (constant) area of an enemy unit without feedback as to the destructiveness of this fire. This situation is shown in Figure 1. X's decision criterion is the net worth* of survivors (with a linear utility for survivors**). For the purposes of this report, discussion of the stopping rule (i.e. battle termination model) is not essential and is omitted.

Mathematically, the problem may be stated as

$$\text{maximize} \quad \{\underset{\sim}{v}^T \underset{\sim}{x}(T) - \underset{\sim}{w}^T \underset{\sim}{y}(T)\},$$

$$\text{subject to:} \quad \frac{dx_1}{dt} = -a_1 y_1, \qquad (1)$$

$$\frac{dx_2}{dt} = -a_2 y_2,$$

---

*Pugh and Mayberry [4] suggest using the ratio of aggregated force value. For our purposes here the exact form of the criterion functional is not essential.

**See [2] for methodology for the development of these linear utilities.

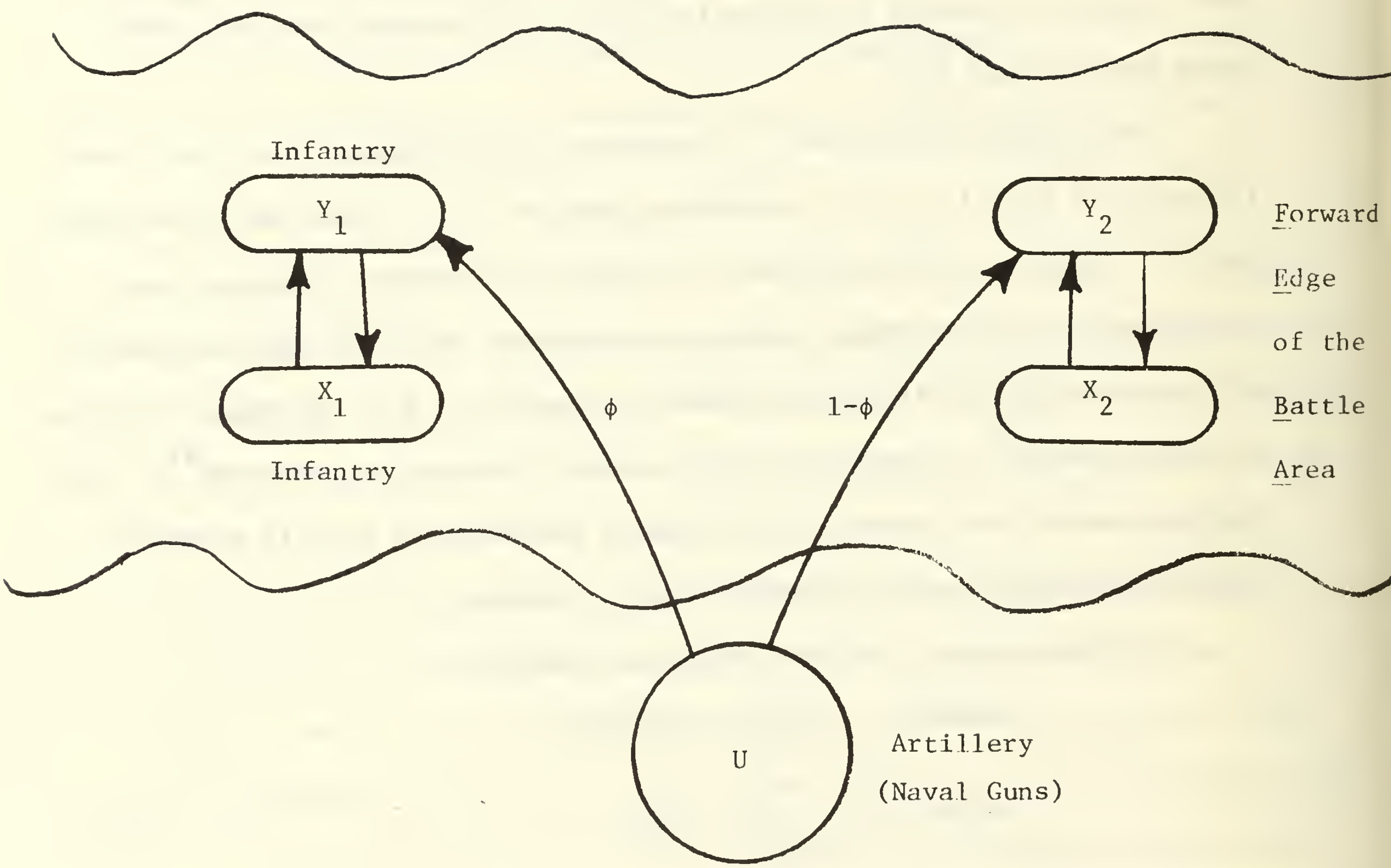

Figure 1. Diagram for Fire Support of Several Ground Units.

$$\frac{dy_1}{dt} = -b_1 x_1 - \phi c_1 y_1,$$

$$\frac{dy_2}{dt} = -b_2 x_2 - (1-\phi) c_2 y_2,$$

with initial conditions

$$x_i(t=0) = x_i^o \quad \text{and} \quad y_i(t=0) = y_i^o \quad \text{for} \quad i = 1,2,$$

and

$x_1, x_2, y_1, y_2 \geq 0$ (State Variable Inequality Constraints),

$0 \leq \phi \leq 1$ (Control Variable Inequality Constraints),

where

$x_i(t)$ [for $i = 1,2$] is the number of X_i infantry at time t,

similarly for $y_i(t)$,

a_i [for $i = 1,2$] is the constant (Lanchester) attrition-rate coefficient (the rate at which one Y_i unit can destroy X_i),

similarly for b_i,

c_i [for $i = 1,2$] is the constant (Lanchester) attrition-rate coefficient for the effectiveness (area fire) of U artillery against Y_i,

$\underset{\sim}{v}^T = (v_1, v_2)$,

v_i is the value of one surviving X_i unit,

similarly for $\underset{\sim}{w}$.

To be precise, in the above formulation we have

$$\frac{dx_i}{dt} = \begin{cases} -a_i y_i & \text{for} \quad x_i > 0, \\ 0 & \text{for} \quad x_i = 0, \end{cases}$$

and

$$\frac{dy_1}{dt} = \begin{cases} -b_1x_1 - \phi c_1y_1 & \text{for } y_1 > 0, \\ 0 & \text{for } y_1 = 0. \end{cases}$$

Moreover, we would begin by considering the case in which x_1^o, x_2^o, y_1^o, and y_2^o are such that $x_1(T), x_2(T), y_1(T), y_2(T) > 0$.

It should be clear that the above problem (1) is important because it considers a basic allocation problem for artillery fire. With minor modifications the model (1) may be extended to other cases of interest: (a) let $b_1 = b_2 = 0$ for a prescribed duration battle attack scenario (see Appendix D), (b) for an amphibious assault one adds replacements for the X_1 and X_2 forces.

Instead of being called "artillery," the fire support units U may be called "Naval guns." Of particular relevance to Navy problems in the determination of the optimal allocation of Naval gunfire in amphibious assaults.

Finally, it seems appropriate to discuss a technical difficulty of solving the above problem (1). As are <u>all</u> Lanchester-type fire distribution problems, the above problem (1) is a singluar problem of optimal control, since $\frac{\partial^2 H}{\partial \phi^2} \equiv 0$ (see [7]), where H denotes the Hamiltonian. The problem becomes particularly difficult when a singular solution exists in $\underset{\sim}{x} - \underset{\sim}{p}$ space (see [1]). A preliminary investigation has shown that this is the case for the problem at hand. It should be pointed out that we have encountered such difficulty previously in our research reported here (see Appendix D).

3. <u>A Lanchester-Type Optimal Control Problem with Logistics Considerations.</u>

In this section we will describe a simple model which, nevertheless, can generate some insights into tactics used by General George S. Patton, Jr., in World War II. Consider combat between X and Y forces. Part of the X forces (denoted as X_2) can be kept in reserve and consequently do not consume supplies as rapidly as the combatant forces (denoted as X_1) do. It is assumed that the effectiveness of X_1 against Y depends upon the amount of supplies (denoted as S) that they have. For the problem under consideration, this effectiveness will be represented by a Lanchester attrition-rate coefficient $b_1 = b_1(S)$. It seems reasonable to hypothesize that $b_1(S)$ is a concave function of S with certain other appropriate properties. Further research should be done on this important topic. For convenience in our initial investigation, we would assume that b_1 is constant but that $S \geq 0$. We further assume that due to "pipeline" capacity there is an upper limit to the rate at which supplies can be replenished.

The decision variable under the control of the X commander is the rate of reinforcing (or withdrawing for $u(t)$ negative) the X_1 forces. This situation is shown in Figure 4. The objective of the X commander is to maximize the net worth of survivors (considering linear utilities).

In mathematical terms, the problem may be stated as follows.

$$\underset{u(t)}{\text{maximize}}\ \{px_1(T) + qx_2(T) - ry(T)\} \quad \text{with } T \text{ unspecified,}$$

$$\text{subject to:} \qquad \frac{dx_1}{dt} = -ay + u, \tag{2}$$

$$\frac{dx_2}{dt} = -u,$$

$$\frac{dy}{dt} = -bx_1,$$

$$\frac{dS}{dt} = -cx_1 + P,$$

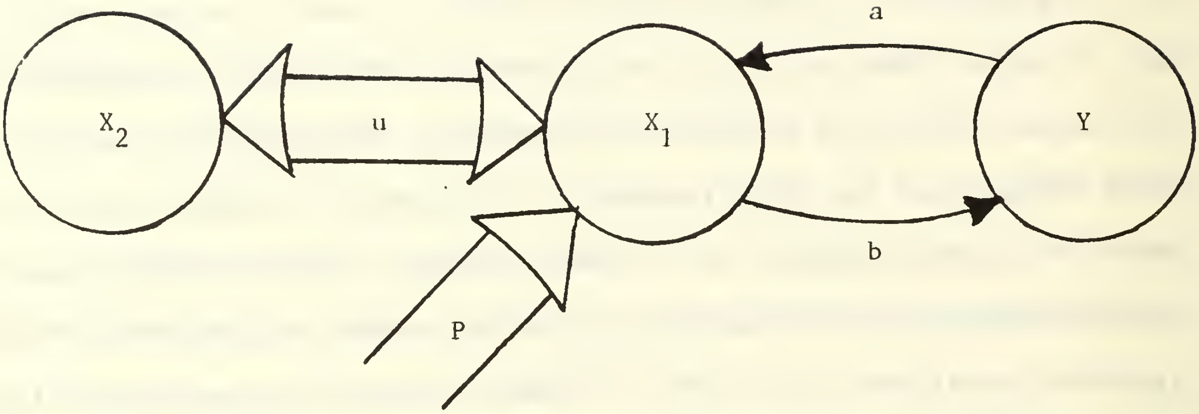

Figure 2. Diagram of Supply-Constrained Combat Problem.

with initial conditions

$$x_i(t=0) = x_i^0 \quad \text{for} \quad i = 1,2, \quad y(t=0) = y_0, \quad S(t=0) = S_0,$$

and

$x_1, x_2, y, S \geq 0$ (State Variable Inequality Constraints),

$-W \leq u \leq R$ (Control Variable Inequality Constraints),

and with the stopping rule

(a) $x_1(T) = 0$,

(b) $y(T) = 0$,

or (c) $S < 0$,

where

c denotes the rate of consumption of supplies S by one X_1 combatant,

$R, W > 0$,

P denotes the "pipeline" capacity,

S denotes the supply level of the X_1 forces,

$u(t)$ denotes the rate of reinforcement of X_1 forces by the X_2 forces (negative quantity denotes withdrawal),

and other notation is similar to that used in the statement of problem (1).

A preliminary examination of problem (2) has yielded an interesting result: if supplies are to become a constraining factor for the X forces later in combat, then the optimal tactic is to "overcommit" forces early in the campaign (i.e. forces are being withdrawn at the moment that the supply constraint becomes active). Of particular mathematical difficulty is the presence of a second order SVIC (i.e. $S \geq 0$) (see [8]) in this problem.

The only previous work considering logistics aspects apparently is by Moglewer and C. Payne [3]. Our above problem (2) may be extended to yield insights for policy planning regarding "pipeline capacity" (logistics capacity), pipeline protection, pipeline interdiction, etc.

4. Form of the Criterion Functional in Time-Sequential Combat Optimization Problems.

In all our past research, we have with one exception (see Appendix D of this report) always considered a linear utility for survivors* in the criterion functional. It is of interest to examine how the valuation of survivors affects the structure of optimal strategies. It seems appropriate to begin such an investigation by considering the simplest problem possible. Thus, we will consider a version of the "Tactical Air-War Campaign" (see [10] and Appendix B of this report).

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*See [2] for methodology for the development of such a linear utility.

Mathematically, the problem may be stated as follows.

$$\underset{\phi(t)}{\text{maximize}} \ \{Q(x(t_f),y(t_f)) + \int_0^{t_f} [x(1-\phi)-y]dt\},$$

with stopping rule: $t_f - T = 0$,

subject to: (air-battle dynamics) $\frac{dx}{dt} = r$,

$$\frac{dy}{dt} = s - \phi b x,$$

with initial conditions

$$x(t=0) = x_0 \quad \text{and} \quad y(t=0) = y_0,$$

and

$x,y \geq 0$ (State Variable Inequality Constraints),

$0 \leq \phi \leq 1$ (Control Variable Inequality Constraints),

where

$x(t)$ and $y(t)$ denote the numbers of X and Y aircraft, respectively, at time t,

b denotes a Lanchester attrition-rate coefficient (the rate at which one X aircraft shoots down Y aircraft),

r and s denote replacement rates,

$\phi(t)$ denotes the fraction (at time t) of total X aircraft which fly counter-air missions and hence shoot down Y aircraft,

and $Q(x(t_f),y(t_f))$ denotes the (salvage) value of X and Y aircraft at the end of the planning horizon.

It is of interest to examine the dependence of the structure of the optimal allocation policy for X aircraft upon the nature of $G(x_f,y_f)$, where x_f denotes $x(t=t_f)$, etc. It seems reasonable on military grounds to require that

$$\frac{\partial Q}{\partial x_f} > 0 \quad \text{and} \quad \frac{\partial Q}{\partial y_f} < 0 \quad \text{for all} \quad x_f, y_f \geq 0.$$

Furthermore, one might postulate either of the following functional forms for $Q(x_f, y_f)$:*

$$\text{(a)} \qquad Q(x_f, y_f) = F(x_f) - G(y_f),$$

or

$$\text{(b)} \qquad Q(x_f, y_f) = \frac{F(x_f)}{G(y_f)} .$$

It is of interest to study cases in which F (and/or G) is a

(A) concave function,

or (B) convex function,

or (C) quasi-concave function,

or (D) quasi-convex function.

To consider a concrete example, if $G(y_f)$ were a concave function, it might look like that shown in Figure 3. One possibility for an analytic representation of the function shown in Figure 3 is

$$G(y_f) = \frac{\beta_0}{\alpha} (1 - e^{-\alpha y_f}). \tag{4}$$

After the dependence of the structure of the optimal allocation policy upon the nature of the terminal return $G(x_f, y_f)$ has been studied for the above problem (3) (preliminary analysis indicates that this problem is analytically tractable), it would seem appropriate to consider a problem like the Isbell-Marlow fire distribution problem (see [5], [6], [8], or [10]), which is at the next level of analytical complexity. Such a research program

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*Pugh and Mayberry [4] have suggested using the ratio of aggregated force values.

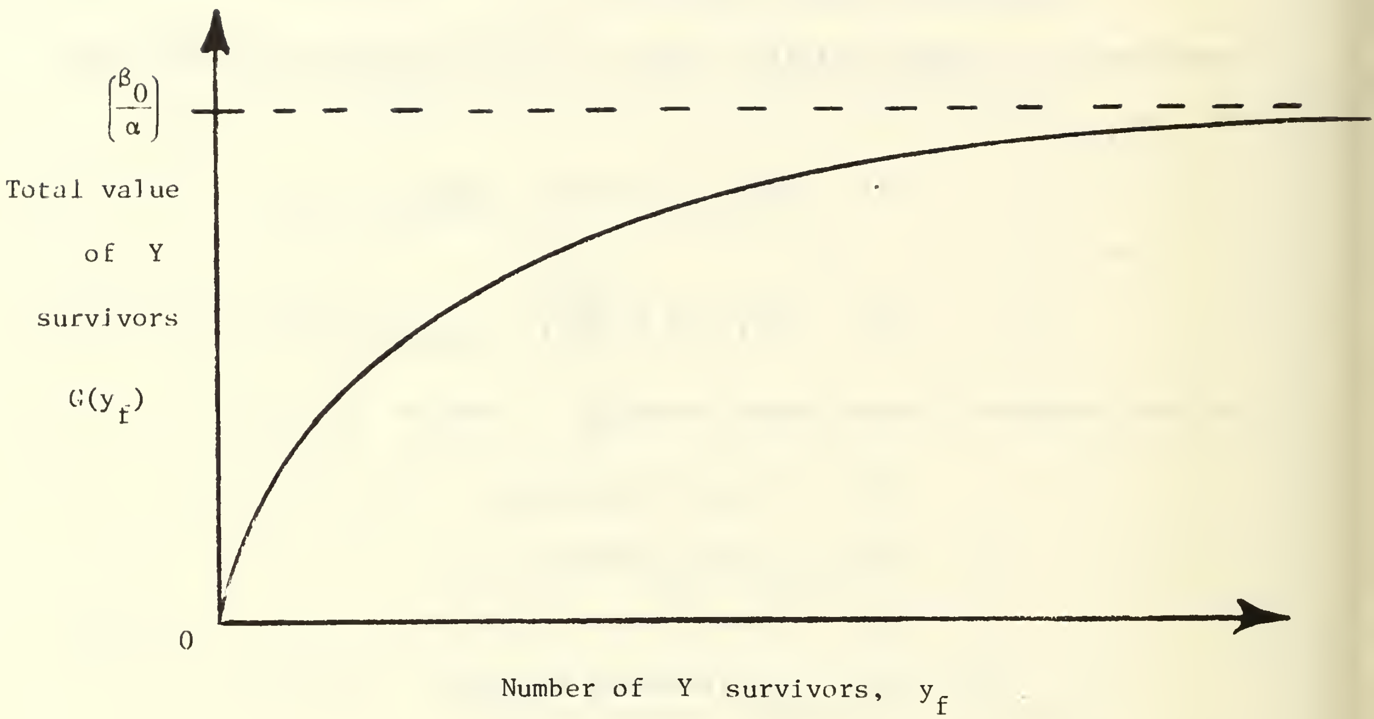

Figure 3. Decreasing Marginal Value for Survivors.

would lead to a better understanding of the effects upon (optimal) military decision making of the valuation of (system) objectives. This in turn would result in a better understanding of the optimization of combat dynamics.

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